3-rd Italian Mathematical Olympiad 1987

Viareggio, April 25, 1987

1. Show that $3x^5 + 5x^3 - 8x$ is divisible by 120 for any integer x.

2. A tetrahedron has the property that the three segments connecting the pairs of midpoints of opposite edges are equal and mutually orthogonal. Prove that this tetrahedron is regular.

3. Show how to construct (by a ruler and a compass) a right-angled triangle, given its inradius and circumradius.

4. Given $I_0 = \{-1, 1\}$, define I_n recurrently as the set of solutions x of the equations

$$x^2 - 2xy + y^2 - 4^n = 0,$$

   where y ranges over all elements of I_{n-1}. Determine the union of the sets I_n over all nonnegative integers n.

5. Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be two arbitrary infinite sequences of natural numbers. Prove that there exist different indices r and s such that $a_r \geq a_s$ and $b_r \geq b_s$.

6. There are three balls of distinct colors in a bag. We repeatedly draw out the balls one by one; the balls are put back into the bag after each drawing. What is the probability that, after n drawings,

   (a) exactly one color occured?

   (b) exactly two oclors occured?

   (c) all three colors occured?

7. A square paper of side n is divided into n^2 unit square cells. A maze is drawn on the paper with unit walls between some cells in such a way that one can reach every cell from every other cell not crossing any wall. Find, in terms of n, the largest possible total length of the walls.

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