6-th Italian Mathematical Olympiad 1990

Cesenatico, May 1990

1. A cube of edge length 3 consists of 27 unit cubes. Find the number of lines passing through exactly three centers of these 27 cubes, as well as the number of those passing through exactly two such centers.

2. In a triangle ABC, the bisectors of the angles at B and A meet the opposite sides at P and Q, respectively. Suppose that the circumcircle of triangle PQC passes through the incenter R of $\triangle ABC$. Given that $PQ = l$, find all sides of triangle PQR.

3. Let a, b, c be distinct real numbers and $P(x)$ a polynomial with real coefficients. Suppose that the remainders of $P(x)$ upon division by $(x-a)$, $(x-b)$ and $(x-c)$ are a, b and c, respectively. Find the polynomial that is obtained as the remainder of $P(x)$ upon division by $(x-a)(x-b)(x-c)$.

4. Let a, b, c be side lengths of a triangle with $a+b+c=1$. Prove that
$$a^2+b^2+c^2+4abc \le \frac{1}{2}.$$

5. Prove that, for any integer x, $x^2+5x+16$ is not divisible by 169.

6. Some marbles are distributed over $2n+1$ bags. Suppose that, whichever bag is removed, it is possible to divide the remaining bags into two groups of n bags such that the number of marbles in each group is the same. Prove that all the bags contain the same number of marbles.