8-th Italian Mathematical Olympiad 1992

Cesenatico, May 1992

1. A cube is divided into 27 equal smaller cubes. A plane intersects the cube. Find the maximum possible number of smaller cubes the plane can intersect.

2. A convex quadrilateral of area 1 is given. Prove that there exist four points in the interior or on the sides of the quadrilateral such that each triangle with the vertices in three of these four points has an area greater than or equal to $1/4$.

3. Prove that for each $n \geq 3$ there exist n distinct positive divisors $d_1, d_2, \ldots, d_n$ of $n!$ such that $n! = d_1 + d_2 + \cdots + d_n$.

4. A jury of 9 persons should decide whether a verdict is guilty or not. Each juror votes independently with the probability 1/2 for each of the two possibilities, and noone is allowed to be abstinent. Find the probability that a fixed juror will be a part of the majority. In the case of a jury of n persons, find the values of n for which the probability of being a part of the majority is greater than, equal to, and smaller than $1/2$, respectively. (For $n = 2k$, $k + 1$ votes are needed for a majority.)

5. Let a, b, c be real numbers. Prove that the smallest of the numbers $(a - b)^2$, $(b - c)^2$, $(c - a)^2$ does not exceed $\dfrac{a^2 + b^2 + c^2}{2}$.

6. Let a and b be integers. Prove that if $\sqrt[3]{a} + \sqrt[3]{b}$ is a rational number, then both a and b are perfect cubes.