9-th Italian Mathematical Olympiad 1993

April 29, 1993

1. Let be given points A,B,C on a line, with C between A and B. Three semicircles with diameters AC, BC, AB are drawn on the same side of line ABC. The perpendicular to AB at C meets the circle with diameter AB at H. Given that $CH = \sqrt{2}$, compute the area of the region bounded by the three semicircles.

2. Find all pairs (p,q) of positive primes such that the equation $3x^2 - px + q = 0$ has two distinct rational roots.

3. Consider an infinite chessboard whose rows and columns are indexed by positive integers. At most one coin can be put on any cell of the chessboard. Let be given two arbitrary sequences (a_n) and (b_n) of positive integers ($n \in \mathbb{N}$). Assuming that infinitely many coins are available, prove that they can be arranged on the chessboard so that there are a_n coins in the n-th row and b_n coins in the n-th column for all n.

4. Let P be a point in the plane of a triangle ABC, different from its circumcenter. Prove that the triangle whose vertices are the projections of P on the perpendicular bisectors of the sides of ABC, is similar to ABC.

5. Prove the following inequality for any positive real numbers a,b,c not exceeding 1:
$$a^2 + b^2 + c^2 \le a^2 b + b^2 c + c^2 a + 1.$$

6. A unit cube $\mathscr{C}$ is rotated around one of its diagonals for the angle $\pi/3$ to form a cube $\mathscr{C}'$. Find the volume of the intersection of $\mathscr{C}$ and $\mathscr{C}'$.

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