Italian IMO Team Selection Test 2000

Cortona, May 2000

Time allowed: 4 hours

1. Determine all triples (x,y,z) of positive integers such that
$$\frac{13}{x^2}+\frac{1996}{y^2}=\frac{z}{1997}\,.$$

2. Let ABC be an isosceles right triangle and M be the midpoint of its hypotenuse AB. Points D and E are taken on the legs AC and BC respectively such that $AD=2DC$ and $BE=2EC$. Lines AE and DM intersect at F. Show that FC bisects the angle $\angle DFE$.

3. Given positive numbers a_1 and b_1, consider the sequences defined by
$$a_{n+1}=a_n+\frac{1}{b_n},\quad b_{n+1}=b_n+\frac{1}{a_n}\quad (n\geq 1).$$
Prove that $a_{25}+b_{25}\geq 10\sqrt{2}$.

4. On a mathematical competition n problems were given. The final results showed that:
   (i) on each problem, exactly three contestants scored 7 points;
   (ii) for each pair of problems, exactly one contestant scored 7 points on both problems.

   Prove that if $n\geq 8$, then there is a contestant who got 7 points on each problem. Is this statement necessarily true if $n=7$?