Italian IMO Team Selection Test 2001

Cortona, May 2001

Time allowed: 4 hours

1. The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at point M. The bisector of $\angle ACD$ meets the ray BA at K. Given that $MA \cdot MC + MA \cdot CD = MB \cdot MD$, prove that $\angle BKC = \angle CDB$.

2. Let $0 \leq a \leq b \leq c$ be real numbers. Prove that

$$(a+3b)(b+4c)(c+2a) \geq 60abc.$$

3. Find all pairs (p,q) of prime numbers such that p divides $5^q + 1$ and q divides $5^p + 1$.

4. We are given 2001 balloons and a positive integer k. Each balloon has been blown up to a size that may depend on a balloon. In each step it is allowed to choose at most k balloons and equalize their sizes to their arithmetic mean. Determine the smallest value of k such that, whatever the initial sizes are, it is possible to make all the balloons have equal size after a finite number of steps.