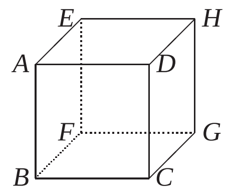
Italian IMO Team Selection Test 2004

First Day – Pisa

1. At the vertices A,B,C,D,E,F,G,H of a cube, $2001,2002,2003,2004,2005,2008,2007$ and 2006 stones respectively are placed. It is allowed to move a stone from a vertex to each of its three neighbors, or to



move a stone to a vertex from each of its three neighbors. Which of the following arrangements of stones at $A,B,\dots,H$ can be obtained?

   (a) $2001,2002,2003,2004,2006,2007,2008,2005$;

   (b) $2002,2003,2004,2001,2006,2005,2008,2007$;

   (c) $2004,2002,2003,2001,2005,2008,2007,2006$.

2. Let $\mathscr{P}_0 = A_0A_1\dots A_{n-1}$ be a convex polygon such that $A_iA_{i+1} = 2^{[i/2]}$ for $i = 0,1,\dots,n-1$ (where $A_n = A_0$). Define the sequence of polygons $P_k = A_0^kA_1^k\dots A_{n-1}^k$ as follows: A_i^1 is symmetric to A_i with respect to A_0, A_i^2 is symmetric to A_i^1 with respect to A_1^1, A_i^3 is symmetric to A_i^2 with respect to A_2^2 and so on. Find the values of n for which infinitely many polygons $\mathscr{P}_k$ coincide with $\mathscr{P}_0$.

3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m,n \in \mathbb{N}$,
$$(2^m+1)f(n)f(2^mn) = 2^mf(n)^2 + f(2^mn)^2 + (2^m-1)^2n.$$

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4. Two circles γ_1 and γ_2 intersect at A and B. A line r through B meets γ_1 at C and γ_2 at D so that B is between C and D. Let s be the line parallel to AD which is tangent to γ_1 at E, at the smaller distance from AD. Line EA meets γ_2 in F. Let t be the tangent to γ_2 at F.

   (a) Prove that t is parallel to AC.

   (b) Prove that lines r,s,t are concurrent.

5. A positive integer n is said to be a *perfect power* if $n = a^b$ for some integers a,b with $b > 1$.

   (a) Find 2004 perfect powers which form an arithmetic progression.

   (b) Prove that perfect powers cannot form an infinite arithmetic progression.

6. Given real numbers x_i, y_i ($i = 1,2,\dots,n$), let A be the $n\times n$ matrix given by $a_{ij} = 1$ if $x_i \geq y_j$ and $a_{ij} = 0$ otherwise. Suppose B is a $n\times n$ matrix whose entries are 0 and 1 such that the sum of entries in any row or column of B equals the sum of entries in the corresponding row or column of A. Prove that $B = A$.

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