Italian IMO Team Selection Test 2005

First Day – Pisa, May 20

1. A stage course is attended by $n \geq 4$ students. The day before the final exam, each group of three students conspire against another student to throw him/her out of the exam. Prove that there is a student against whom there are at least $\sqrt[3]{(n-1)(n-2)}$ conspirators.

2. (a) Prove that in a triangle the sum of the distances from the centroid to the sides is not less than three times the inradius, and find the cases of equality.

   (b) Determine the points in a triangle that minimize the sum of the distances to the sides.

3. The function $\psi : \mathbb{N} \to \mathbb{N}$ is defined by $\psi(n) = \sum_{k=1}^{n} \gcd(k,n)$.

   (a) Prove that $\psi(mn) = \psi(m)\psi(n)$ for every two coprime $m, n \in \mathbb{N}$.

   (b) Prove that for each $a \in \mathbb{N}$ the equation $\psi(x) = ax$ has a solution.

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4. Suppose that $f : \{1,2,\dots,1600\} \to \{1,2,\dots,1600\}$ satisfies $f(1) = 1$ and
$$f^{2005}(x) = x \quad \text{for } x = 1,2,\dots,1600.$$

   (a) Prove that f has a fixed point different from 1.

   (b) Find all $n > 1600$ such that any $f : \{1,\dots,n\} \to \{1,\dots,n\}$ satisfying the above condition has at least two fixed points.

5. The circle Γ and the line ℓ have no common points. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \neq A, B$ is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Let H be the reflection of G in AB. Show that F, C, and H are collinear.

6. Let N be a positive integer. Alberto and Barbara write numbers on a blackboard taking turns, according to the following rules. Alberto starts writing 1, and thereafter if a player has written n on a certain move, his adversary is allowed to write $n+1$ or $2n$ as long as he/she does not obtain a number greater than N. The player who writes N wins.

   (a) Determine which player has a winning strategy for $N = 2005$.

   (b) Determine which player has a winning strategy for $N = 2004$.

   (c) Find for how many integers $N \leq 2005$ Barbara has a winning strategy.

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