Italian IMO Team Selection Test 2006

First Day – Pisa, May 20

1. We consider a string S of 99 letters, 66 of which are A and 33 are B. We call S *good* if, for each initial substring of S (i.e. consisting of the first n letters, $1 \leq n \leq 99$), the number of different words that can be obtained by permuting the letters of the substring is odd. How many good strings are there? Which strings are good?

2. In a triangle ABC, H is the orthocenter and L, M, N the midpoints of the sides AB, BC, CA, respectively. Prove that
$$HL^2 + HM^2 + HN^2 < AL^2 + BM^2 + CN^2$$
if and only if ABC is acite-angled.

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers m, n
$$f(m - n + f(n)) = f(m) + f(n).$$

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4. Circles γ_1 and γ_2 intersect at points Q and R and internally touch a circle γ at A_1 and A_2 respectively. Let P be an arbitrary point on γ. Segments PA_1 and PA_2 meet γ_1 and γ_2 again at B_1 and B_2, respectively.
   (a) Prove that the tangent to γ_1 at B_1 and the tangent to γ_2 at B_2 are parallel.
   (b) Prove that B_1B_2 is a common tangent of γ_1 and γ_2 if and only if P lies on QR.

5. For each positive integer n, let A_n denote the set of positive integers $a \leq n$ such that $n \mid a^n + 1$.
   (a) Find all n for which A_n is nonempty.
   (b) Find all n for which $|A_n|$ is even and nonzero.
   (c) Is there an n with $|A_n| = 130$?

6. Let $p(x)$ be a complex polynomial with $p(0) \neq 0$. Show that there exists a polynomial with positive real coefficients divisible by $p(x)$ if and only if $p(x)$ has no positive real root.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com