Italian IMO Team Selection Test 1998

Cortona, May 23, 1998

Time allowed: 4 hours

1. Given a real number α, find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\alpha x^2 f\left(\frac{1}{x}\right) + f(x) = \frac{x}{x+1} \quad \text{for all } x \in \mathbb{R}^+.$$

2. In a triangle ABC, points H, M, L are the feet of the altitude from C, the median from A, and the angle bisector from B, respectively. Show that if triangle HML is equilateral, then so is triangle ABC.

3. New license plates consist of two letters, three digits, and two letters (from the English alphabet of 26 letters). What is the largest possible number of such license plates if it is required that every two of them differ at no less than two positions?

4. Find all polynomials $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ whose zeros (with their multiplicities) are exactly $a_1, a_2, \ldots, a_n$.