51-st Polish Mathematical Olympiad 1999/2000

First Round

September – December, 1999

1. Prove that, for every integer $n \geq 3$, the sum of the cubes of all natural numbers less than n and coprime with n is divisible by n.

2. In an acute-angled triangle ABC with $\angle ACB = 2\angle ABC$, D is the point on side BC satisfying $2\angle BAD = \angle ABC$. Prove that
$$\frac{1}{BD} = \frac{1}{AB} + \frac{1}{AC}.$$

3. The sum of positive numbers a, b, c is 1. Prove that $a^2 + b^2 + c^2 + 2\sqrt{3abc} \leq 1$.

4. Each point of a circle is painted with one of three colors. Prove that there exist three points of the same color on the circle which are vertices of an isosceles triangle.

5. Find all pairs (a, b) of positive integers such that the numbers $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are cubes of positive integers.

6. A point X lies inside or on the boundary of the triangle ABC with $\angle C = 90^\circ$. Points P, Q, R are the projections of X onto BC, CA, and AB respectively. Prove that the equality $AR \cdot RB = BP \cdot PC + AQ \cdot QC$ holds if and only if X lies on the side AB.

7. Show that for each positive integer n and each number $t \in \left(\frac{1}{2}, 1\right)$ there exist numbers $a, b \in (1999, 2000)$, such that
$$\frac{1}{2}a^n + \frac{1}{2}b^n < (ta + (1-t)b)^n.$$

8. The numbers $c(n, k)$ are defined for integers $n \geq k \geq 0$ by $c(n, 0) = c(n, n) = 1$ for all $n \geq 0$ and
$$c(n+1, k) = 2^k c(n, k) + c(n, k-1) \quad \text{for} \quad n \geq k \geq 1.$$
Prove that $c(n, k) = c(n, n-k)$ for all n and k.

9. Suppose that positive integers m and n are such that mn divides $m^2 + n^2 + m$. Prove that m is a perfect square.

10. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be pairwise orthogonal unit vectors in space. Let ω be a variable plane through O, and let A', B', C' be the projections of A, B, C onto ω. Find the set of values of $OA'^2 + OB'^2 + OC'^2$ when ω takes all possible positions.

11. Let M be a set of n^2+1 positive integers having the following property: in each $n+1$ numbers from M there are two numbers, one of which divides the other. Prove that there are different elements $a_1,\dots,a_{n+1}$ of M such that $a_{i+1} \mid a_i$ for $i=1,2,\dots,n$.

12. Points D,E,F are taken on the respective sides BC,CA,AB of an acute-angled triangle ABC. The circumcircles of the triangles AEF, BFD, CDE meet at point P. Prove that if

$$\frac{PD}{PE}=\frac{BD}{AE},\quad \frac{PE}{PF}=\frac{CE}{BF},\quad \frac{PF}{PD}=\frac{AF}{CD},$$

then AD, BE, CF are the altitudes of triangle ABC.

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