56-th Polish Mathematical Olympiad 2004/05

First Round

September 11 – December 10, 2004

1. Solve in real numbers the system

$$\begin{aligned} x^2 &= yz+1 \\ y^2 &= zx+2 \\ z^2 &= xy+4. \end{aligned}$$

2. Find all integers $n>1$ for which $2^2+3^2+\dots+n^2$ is a power of a prime.

3. In an acute-angled ABC point D is the projection of C onto AB, and E is the projection of D onto BC. Point F is taken on the segment DE so that $EF:FD = AD:DB$. Prove that the lines CF and AE are perpendicular.

4. A natural number n and positive real numbers a and b are given. Find the largest possible value of the expression

$$x_1y_1+x_2y_2+\dots+x_ny_n,$$

where x_i, y_i are numbers from the interval $[0,1]$ such that $x_1+x_2+\dots+x_n \le a$ and $y_1+y_2+\dots+y_n \le b$.

5. A quadrilateral $ABCD$ is inscribed in a circle, and the incircles of the triangles ABC and BCD have equal radii. Prove that the incircles of the triangles CDA and DAB have equal radii as well.

6. Determine whether there exists an infinite sequence $a_1, a_2, \dots$ of positive integers satisfying $\dfrac{1}{a_n} = \dfrac{1}{a_{n+1}} + \dfrac{1}{a_{n+2}}$ for all $n \in \mathbb{N}$.

7. Three spheres are pairwise externally tangent and touch a plane at points A, B, C. Given that $BC=a$, $CA=b$, $AB=c$, find the radii of the spheres.

8. On a circle are given n lamps, each of which can be either turned on or turned off. A sequence of operations is performed: in every operation one selects k successive lamps and changes the state of each of them. Initially all the lamps are turned off. For a given positive integer n, find all positive integers $k \le n$ for which one can have all the lamps turned on.

9. Determine all real numbers a such that the sequence (x_n) given by

$$x_0=\sqrt{3}, \quad x_{n+1} = \frac{1+ax_n}{a-x_n} \text{ for } n=0,1,2,\dots$$

satisfies the condition $x_{n+8}=x_n$ for all $n \ge 0$.

10. Three subsets A, B, C of a given n-element set X have been chosen at random. It is assumed that each of the 2^n subsets of X is equiprobable. Find the most probable number of elements of the set $A \cap B \cap C$.

11. A circle with center I is inscribed in a convex quadrilateral $ABCD$, where I does not lie on AC. The diagonals AC and BD intersect at E. The line through E perpendicular to BD meets the lines AI and CI at P and Q, respectively. Prove that $PE = EQ$.

12. Consider the functions $f(x) = 2^x$ and $g(x) = f(f(f(f(f(f(f(x)))))))$ (the seventh iteration of f). Show that the number $g(3) - g(0)$ is divisible by $g(2) - g(0)$.

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