59-th Polish Mathematical Olympiad 2007/08

First Round

September 10 – December 10, 2007

1. Solve in real numbers x, y, z the system of equations

$$\begin{cases} x^5 = 5y^3 - 4z \\ y^5 = 5z^3 - 4x \\ z^5 = 5x^3 - 4y \end{cases}$$

2. Inside a convex angle with vertex P is given a point A. Points X and Y lie on different rays of the angle so that $PX = PY$ and the sum $AX + AY$ is minimal. Prove that $\angle XAP = \angle YAP$.

3. A sequence (a_n) of integers is defined by $a_1 = 1$, $a_2 = 2$ and

$$a_n = 3a_{n-1} + 5a_{n-2} \quad \text{for } n = 3, 4, 5, \ldots$$

Does there exist an integer $k \geq 2$ for which a_k divides $a_{k+1}a_{k+2}$.

4. Let $n \geq 1$ be a given integer. For each nonempty subset A of $\{1, 2, \ldots, n\}$ define the number $w(A)$ as follows: If $a_1 > a_2 > \cdots > a_k$ are the elements of A, then $w(A) = a_1 - a_2 + a_3 - \cdots + (-1)^{k+1}a_k$. Find the sum of the numbers $w(A)$ over all $2^n - 1$ possible subsets A.

5. Find all triples (p, q, r) of prime numbers for which

$$pq + qr + rp \quad \text{and} \quad p^3 + q^3 + r^3 - 2pqr$$

are divisible by $p + q + r$.

6. Find all polynomials $W(x)$ with real coefficients such that $W(x^2)W(x^3) = W(x)^5$ holds for every real number x.

7. In a set of n people, each of its $2^n - 1$ nonempty subsets is called a *company*. Each company should elect a leader, according to the following rule: If a company C is the union $A \cup B$ of two companies A and B, then the leader of C is also the leader of at least one of the companies A and B. Find the number of possible choices of leaders.

8. The base of a pyramid $SABCD$ is a convex quadrilateral $ABCD$. A sphere is inscribed in the pyramid and touches the base $ABCD$ at point P. Prove that $\angle APB + \angle CPD = 180^\circ$.

9. Determine the smallest real number a having the following property: For any real numbers $x, y, z \geq a$ satisfying $x + y + z = 3$, it holds that $x^3 + y^3 + z^3 \geq 3$.

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10. A prime number p is given. A sequence of positive integers $a_1, a_2, \ldots$ satisfies the relation

$$a_{n+1} = a_n + p\left[\sqrt[p]{a_n}\right] \quad \text{for } n = 1,2,3\ldots$$

Show that there is a term in this sequence which is the p-th power of an integer.

11. Points $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ respectively lie on the sides BC, CA, AB, BC, CA, AB, BC respectively of a triangle ABC, and satisfy

$$\angle P_1P_2C = \angle AP_2P_3 = \angle P_3P_4B = \angle CP_4P_5 = \angle P_5P_6A = \angle BP_6P_7 = 60^\circ.$$

Prove that $P_1 \equiv P_7$.

12. Let be given an integer $m \geq 2$. Find the smallest integer $n \geq m$ with the property that, for every partition of the set $\{m, m+1, \ldots, n\}$ into two subsets, one of the subsets contains three numbers a, b, c (not necessarily distinct) with $ab = c$.

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