51-st Polish Mathematical Olympiad 1999/2000

Second Round

February 25–26, 2000

First Day

1. Prove or disprove that every positive rational number can be written in the form $\dfrac{a^2+b^3}{c^5+d^7}$, where a,b,c,d are positive integers.

2. In the triangle ABC the bisector of the angle $\angle BAC$ meets the circumcircle of $\triangle ABC$ at the point $D \neq A$. If K and L are the projections of B and C onto line AC, respectively, show that $AD \geq BK+CL$.

3. In the cells of the $n \times n$ board are written n^2 different positive integers. In each column of the chessboard the cell with the greatest number is colored red. A set S of n cells is called *admissible* if no two cells from S lie in the same column or row. Prove that the admissible set of cells with the greatest sum of numbers contains at least one red cell.

Second Day

4. In a triangle ABC with $AB \neq AC$, I is the incenter and D and E the intersection points of BI and CI with the opposite sides of the triangle, respectively. Find all possible measures of $\angle BAC$ for which the equality $DI = EI$ can be satisfied.

5. Prove or disprove that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(n)) = 2n \quad \text{for all } n \in \mathbb{N}.$$

6. Let w be a quadratic polynomial with integer coefficients. Suppose that for each integer x the value $w(x)$ is a perfect square. Prove that w is the square of a polynomial.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com