54-th Polish Mathematical Olympiad 2002/03

Second Round

February 21–22, 2003

First Day

1. Show that there exists a positive integer $n > 2003$ for which the sequence $a_k = \binom{n}{k}$, $k = 0, 1, \dots, 2003$, has the following property: a_k divides a_m whenever $0 \le k < m \le 2003$.

2. A quadrilateral $ABCD$ is inscribed in a circle o. The bisectors of the angles DAB and ABC meet at P, and the bisectors of the angles BCD and CDA meet at Q. Let M be the midpoint of the arc BC of o not containing D and A, and N be the midpoint of the arc DA of o not containing B and C. Prove that the line through P and Q is perpendicular to MN.

3. Consider the polynomial $W(x) = x^4 - 3x^3 + 5x^2 - 9x$. Find all pairs of distinct integers a, b satisfying $W(a) = W(b)$.

Second Day

4. Show that for each prime $p > 3$ there exist integers x, y, k with $0 < 2k < p$ such that $kp + 3 = x^2 + y^2$.

5. From a point A exterior to a circle o with center O, two tangent lines touching o at B and C are drawn. Another line tangent to o intersects the segments AB and AC at points E and F, respectively. The lines OE and OF respectively meet segment BC at P and Q. Prove that segments BP, PQ, and QC are sides of a triangle similar to $\triangle AEF$.

6. A function f from the pairs of nonnegative integers to the real numbers satisfies the following conditions:

$$f(0,0) = 0, \qquad f(2x,2y) = f(2x+1,2y+1) = f(x,y),$$
$$f(2x+1,2y) = f(2x,2y+1) = f(x,y) + 1$$

for all nonnegative integers x, y. Let n and a, b be nonnegative integers such that $f(a,b) = n$. Find the number of integral solutions x of the equation $f(a,x) + f(b,x) = n$.

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