59-th Polish Mathematical Olympiad 2007/08

Second Round

February 22–23, 2008

First Day

1. Find the largest possible length of a sequence of consecutive integers which are all expressible in the form $x^3 + 2y^2$ for some integers x, y.

2. A convex pentagon $ABCDE$ is such that $\angle ABD = \angle ACE$, $\angle ACB = \angle ACD$, $\angle ADC = \angle ADE$, and $\angle ADB = \angle AEC$. The diagonals BD and CE intersect at S. Prove that AS is perpendicular to CD.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(f(x) - y) = f(x) + f(f(y) - f(-x)) + x \quad \text{for all real } x, y.$$

Second Day

4. In every square of an $n \times n$ board there is an integer such that the sum of all integers in the board is 0. A move consists of choosing a square and decreasing the number in it by the number of neighboring squares (by side), while increasing the numbers in each of the neighboring squares by 1. Determine if there is an $n \geq 2$ for which we can always turn all the integers into zeros in finitely many moves.

5. In a triangle ABC with $AC = BC$, point D on the side AB is such that $AD < DB$ and E is the reflection of A in CD. Prove that

$$\frac{AC}{CD} = \frac{BE}{BD - AD}.$$

6. If n is a positive integer not divisible by 3, show that there exists a positive integer m such that every integer not smaller than m is the sum of digits of some multiple of n.

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