49-th Polish Mathematical Olympiad 1997/98

Second Round

February 27–28, 1998

First Day

1. Let $A_n = \{1, 2, \dots, n\}$. Prove or disprove: For all integers $n \geq 2$ there exist functions $f, g : A_n \to A_n$ which satisfy

$$f(f(k)) = g(g(k)) = k \quad \text{for } 1 \leq k \leq n, \qquad \text{and}$$
$$g(f(k)) = k + 1 \qquad \text{for } 1 \leq k \leq n - 1.$$

2. Let ABC be a triangle with an obtuse angle $\angle C$ and $\angle A = 2\angle B$. The line through B perpendicular to BC intersects line AC at D. If M is the midpoint of AB, prove that $\angle AMC = \angle BMD$.

3. (a) If a, b, c, d, e, f are positive numbers with sum 1 and $ace + bdf \geq \dfrac{1}{108}$, show that

$$abc + bcd + cde + def + efa + fab \leq \frac{1}{36}.$$

   (b) Are there different positive numbers a, b, c, d, e, f with sum 1 for which the two above inequalities become equalities?

Second Day

4. Find all pairs of integers (x, y) satisfying $x^2 + 3y^2 = 1998x$.

5. Suppose that $a_1, \dots, a_7, b_1, \dots, b_7$ are nonnegative real numbers such that $a_i + b_i \leq 2$ for all i. Prove that there are two different indices k, m such that

$$|a_k - a_m| + |b_k - b_m| \leq 1.$$

6. Prove that the edges AB and CD of a tetrahedron $ABCD$ are perpendicular if and only if there exists a parallelogram $CDPQ$ such that $PA = PB = PD$ and $QA = QB = QC$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com