# Romanian IMO Team Selection Tests 2004

*First Test*

Time: 4 hours.

1. Let a quadrilateral have sides $a_1, a_2, a_3, a_4$ and the semiperimeter $s$. Prove the inequality

$$\sum_{i=1}^{4} \frac{1}{s+a_i} \leq \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}.$$

When does equality occur?

2. A finite family of disjoint rectangles of total area 4 is given in the coordinate plane so that the projection of their union onto the horizontal axis is an interval. Prove that there are three points in the union of the rectangles which form a triangle of area at least 1.

3. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$f(f(n)) \leq \frac{n+f(n)}{2}.$$

4. Let $\mathscr{D}$ be a closed disc in the complex plane. Prove that for any $n \in \mathbb{N}$ and complex numbers $z_1, z_2, \ldots, z_n \in \mathscr{D}$ there exists $z \in \mathscr{D}$ such that

$$z^n = z_1 z_2 \ldots z_n.$$

*Second Test*

Time: 4 hours.

1. A disk is divided into $2n$ equal sectors. Among these sectors, $n$ are colored in black and $n$ in white. Let us number the white sectors from 1 to $n$ clockwise, and the black ones from 1 to $n$ counterclockwise. Show that there is a half-disc consisting of $n$ sectors which contains all numbers from 1 to $n$.

2. Find all positive integers that can be written in the form $\dfrac{a^2 + ab + b^2}{ab - 1}$ for some positive integers $a, b$ not both equal to 1.

3. Given integers $a, b, c$ with $b$ odd, define a sequence $(x_n)$ by $x_0 = 4$, $x_1 = 0$, $x_2 = 2c$, $x_3 = 3b$ and
$$x_{n+3} = ax_{n-1} + bx_n + cx_{n+1}.$$

Prove that $p \mid x_{p^m}$ for all primes $p$ and positive integers $m$.

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4. A square $ABCD$ lies inside a circle $\mathscr{C}$. Let $\mathscr{C}_A$ be the circle outside the square that is tangent to $AB, AD$ and $\mathscr{C}$. The circles $\mathscr{C}$ and $\mathscr{C}_A$ meet at $A_1$. Points $B_1, C_1, D_1$ are similarly defined. Show that lines $AA_1, BB_1, CC_1, DD_1$ are concurrent.

*Third Test*

Time: 4 hours.

1. Let $S$ be a finite set with $|S| \geq 2$. Suppose $A_1, A_2, \ldots, A_{101}$ are subsets of $S$ are such that the union of any 50 of them has at least $50|S|/51$ elements. Prove that among these subsets there exist three (distinct), any two of which have a non-empty intersection.

2. Prove that for all positive integers $m, n$ with $m$ odd it holds that

$$3^m n \mid \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

3. In a non-isosceles triangle $ABC$, let $I$ be the incenter and $A', B', C'$ be the projections of $I$ onto $BC, CA, AB$, respectively. Lines $AA', BB', CC'$ meet at point $P$, lines $BC$ and $B'C'$ meet at $N$, and $AC$ and $A'C'$ meet at $M$. Prove that $IP \perp MN$.

4. Let $a_1, a_2, \ldots, a_n$ be real numbers, where $n \geq 2$. Prove that for any subset $S$ of $\{1, 2, \ldots, n\}$ we have

$$\left( \sum_{i \in S} a_i \right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \cdots + a_j)^2.$$

*Fourth Test*

Time: 4 hours.

1. Let $m, n \in \mathbb{N}$, $m > 1$. Suppose that $a^m - 1$ is divisible by $n$ for all $a$ coprime to $n$. Prove that
$$n \leq 4m(2^m - 1).$$

2. Let $O$ be a point in the plane of triangle $ABC$, and $\mathscr{C}$ be a circle passing through $O$. Let $\mathscr{O}$ intersect lines $OA, OB, OC$ at $P, Q, R$, and circles $BOc, COA, AOB$ at $K, L, M$, respectively. Show that the lines $PK, QL$ and $RM$ are concurrent.

3. Faces of a polyhedron with $n$ faces are colored in black and white so that no two black faces have a common point. Prove that the number of edges shared by two white faces is at least $n - 2$.

1. Let be given circles $K_1, K_2, K_3$ of radii $R_1, R_2, R_3$ through a point $O$, and let $A, B, C$ be the other points of intersection of the pairs of circles $(K_2, K_3)$, $(K_3, K_1)$ and $(K_1, K_2)$, respectively. Suppose that $O$ is inside the triangle $ABC$ and let $AO, BO, CO$ meet the opposite sides of the triangle at $A_1, B_1, C_1$, respectively. If $R$ is the circumradius of $ABC$, prove that

$$R \le \frac{OA_1}{AA_1} R_1 + \frac{OB_1}{BB_1} R_2 + \frac{OC_1}{CC_1} R_3.$$

2. An operation of an $m \times n$ chessboard consists of the following:

   (i) choose several unmarked cells, no two in the same row or column;

   (ii) mark these squares with 1;

   (iii) mark with 0 every unmarked square that is in the same row or column with a square marked with 1.

   A game consists of performing these operations as long as it is possible. What is the maximum possible sum of the numbers on the board at the end of a game?

3. Let $p$ be a prime number and $f(x) = \sum_{i=1}^{p-1} a_i x^{i-1}$ be a polynomial with $a_i = 1$ if $i$ is a square modulo $p$ and $a_i = -1$ otherwise.

   (a) Prove that $f(x) \equiv x - 1$ modulo $(x-1)^2$ if $p \equiv 3 \pmod 4$;

   (b) Prove that $f(x) \equiv 0$ modulo $(x-1)^2$ if $p \equiv 5 \pmod 8$.