29-th All-Russian Mathematical Olympiad 2003

Final Round – Gorod Oryol, April 14–20

Grade 9

First Day – April 15

1. Suppose that M is a set of 2003 numbers such that, for any distinct $a,b \in M$, the number $a^2 + b\sqrt{2}$ is rational. Prove that $a\sqrt{2}$ is rational for all $a \in M$. *(N. Agakhanov)*

2. Two circles $\mathscr{S}_1$ and $\mathscr{S}_2$ with centers O_1 and O_2 respectively intersect at A and B. The tangents at A to $\mathscr{S}_1$ and $\mathscr{S}_2$ meet segments BO_2 and BO_1 at K and L respectively. Show that $KL \parallel O_1O_2$. *(S. Berlov)*

3. On a line are given $2k-1$ white segments and $2k-1$ black ones. Assume that each white segment intersects at least k black segments, and each black segment intersects at least k white ones. Prove that there are a black segment intersecting all the white ones, and a white segment intersecting all the black ones. *(V. Dolnikov)*

4. A sequence (a_n) is defined as follows: $a_1 = p$ is a prime number with exactly 300 nonzero digits, and for each $n \geq 1$, a_{n+1} is the decimal period of $1/a_n$ multiplies by 2. Determine a_{2003}. *(I. Bogdanov, A. Hrabrov)*

Second Day – April 16

5. There are N cities in a country. Any two of them are connected either by a road or by an airway. A tourist wants to visit every city exactly once and return to the city at which he started the trip. Prove that he can choose a starting city and make a path, changing means of transportation at most once. *(O. Podlipskiy)*

6. Let a,b,c be positive numbers with the sum 1. Prove the inequality
$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} \geq \frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}.$$
(S. Berlov)

7. Is it possible to write a natural number in every cell of an infinite chessboard in such a manner that for all integers $m,n > 100$, the sum of numbers in every $m \times n$ rectangle is divisible by $m+n$? *(S. Berlov)*

8. Let B and C be arbitrary points on sides AP and PD respectively of an acute triangle APD. The diagonals of the quadrilateral $ABCD$ meet at Q, and H_1, H_2 are the orthocenters of triangles APD and BPC, respectively. Prove that if the line H_1H_2 passes through the intersection point X ($X \neq Q$) of the circumcircles of triangles ABQ and CDQ, then it also passes

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through the intersection point Y ($Y \neq Q$) of the circumcircles of triangles BCQ and ADQ. *(S. Berlov, L. Emelyanov)*

Grade 10

First Day

1. Suppose that M is a set of 2003 numbers such that, for any distinct $a, b, c \in M$, the number $a^2 + bc$ is rational. Prove that there is a natural number n such that $a\sqrt{n}$ is rational for all $a \in M$. *(N. Agakhanov)*

2. The diagonals of a cyclic quadrilateral $ABCD$ meet at O. Let $\mathscr{S}_1, \mathscr{S}_2$ be the circumcircles of triangles ABO and CDO respectively, and O, K their intersection points. The lines through O parallel to AB and CD meet $\mathscr{S}_1$ and $\mathscr{S}_2$ again at L and M, respectively. Points P and Q on segments OL and OM respectively are taken such that $OP : PL = MQ : QO$. Prove that O, K, P, Q lie on a circle. *(S. Berlov)*

3. Let be given a tree (i.e. a connected graph with no cycles) with n vertices. Its vertices are assigned numbers $x_1, x_2, \dots, x_n$, and each edge is assigned the product of the numbers at its endpoints. Let S denote the sum of the numbers at the edges. Prove that
$$2S \leq \sqrt{n-1}(x_1^2 + x_2^2 + \cdots + x_n^2).$$
(V. Dolnikov)

4. A finite set of points X and an equilateral triangle T are given on a plane. Suppose that every subset X' of X with no more than 9 elements can be covered by two images of T under translations. Prove that the whole set X can be covered by two images of T under translations.
(V. Dolnikov, P. Karasev)

Second Day

5. *Problem 5 for Grade 9.*

6. Let a_0 be a natural number. The sequence (a_n) is defined by $a_{n+1} = a_n/5$ is a_n is divisible by 5, and $a_{n+1} = \left[\sqrt{5}a_n\right]$ otherwise. Show that the sequence a_n is increasing starting from some term. *(A. Hrabrov)*

7. In a triangle ABC, O is the circumcenter and I the incenter. The excircle ω_a touches rays AB, AC and side BC at K, M, N, respectively. Prove that if the midpoint P of KM lies on the circumcircle of $\triangle ABC$, then points O, N, I lie on a line.
(P. Kozhevnikov)

8. Find the greatest natural number N such that, for any arrangement of the numbers $1, 2, \dots, 400$ in a chessboard 20×20, there exist two numbers in the same row or column, which differ by at least N. *(D. Hramtsov)*

Grade 11

First Day

1. Let $\alpha, \beta, \gamma, \delta$ be positive numbers such that for all x,

$$\sin \alpha x + \sin \beta x = \sin \gamma x + \sin \delta x.$$

Prove that $\alpha = \gamma$ or $\alpha = \delta$. *(N. Agakhanov, A. Golovanov, V. Senderov)*

2. *Problem 2 for Grade 10.*

3. Let $f(x)$ and $g(x)$ be polynomials with non-negative integer coefficients, and let m be the largest coefficient of f. Suppose that there exist naturalk numbers $a < b$ such that $f(a) = g(a)$ and $f(b) = g(b)$. Show that if $b > m$, then $f \equiv g$. *(A. Hrabrov)*

4. Ana and Bora are each given a sufficiently long paper strip, one with letter A written, and the other with letter B. Every minute, one of them (not necessarily one after another) writes either on the left or on the right to the word on his/her strip the word written on the other strip. Prove that the day after, one will be able to cut the word on Ana's strip into two words and exchange their places, obtaining a palindromic word.

(E. Cherepanov)

Second Day

5. The side lengths of a triangle are the roots of a cubic polynomial with rational coefficients. Prove that the altitudes of this triangle are roots of a polynomial of sixth degree with rational coefficients. *(N. Agakhanov)*

6. Is it possible to write a natural number in every cell of an infinite chessboard in such a manner that for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$? *(S. Berlov)*

7. There are 100 cities in a country, some of them being joined by roads. Any four cities are connected to each other by at least two roads. Assume that there is no path passing through every city exactly once. Prove that there are two cities such that every other city is connected to at least one of them. *(I. Ivanov)*

8. The inscribed sphere of a tetrahedron $ABCD$ touches ABC, ABD, ACD and BCD at D_1, C_1, B_1 and A_1 respectively. Consider the plane equidistant from A and plane $B_1C_1D_1$ (parallel to $B_1C_1D_1$) and the three planes defined analogously for the vertices B, C, D. Prove that the circumcenter of the tetrahedron formed by these four planes coincides with the circumcenter of tetrahedron $ABCD$. *(F. Bakharev)*