# 31-st All-Russian Mathematical Olympiad 2005

## Final Round – Nizhniy Novgorod, April 24–29

### Grade 9

*First Day*

1. A parallelogram $ABCD$ with $AB < BC$ is given. Points $P$ and $Q$ vary on the sides $BC$ and $CD$ respectively such that $CP = CQ$. Show that the circumcircles of all triangles $APQ$ have a common point other than $A$.
*(T. Emelyanova)*

2. Lyosha wrote down the numbers $1, 2, \dots, 22^2$ in the cells of a $22 \times 22$ table using each number exactly once. Can Oleg always choose a pair of cells sharing a side or a vertex such that the sum of the numbers in these cells is divisible by 4? *(O. Podlipsky)*

3. Real numbers $a_1, a_2, a_3 > 1$ satisfy $a_1 + a_2 + a_3 = S$ and $\dfrac{a_i^2}{a_i - 1} > S$ for $i = 1, 2, 3$. Prove that
$$\frac{1}{a_1 + a_2} + \frac{1}{a_2 + a_3} + \frac{1}{a_3 + a_1} > 1.$$
*(S. Berlov)*

4. On the reverse sides of 365 cards on a table, distinct numbers were written. For one ruble, Vasya can choose three cards and ask Petya to arrange the numbers on them in the increasing order from left to right. Can Vasya always arrange the numbers on the cards in the increasing order from left to right, paying 2000 rubles? *(M. Garber)*

*Second Day*

5. Ten distinct nonzero numbers are such that for any two of these numbers, either their sum or their product is rational. Prove that the squares of all these numbers are rational. *(O. Podlipsky)*

6. Let $S(M)$ denotes the sum of the elements of a set $M$. In how many ways can one partition the numbers $2^0, 2^1, 2^2, \dots, 2^{2005}$ into two nonempty subsets $A$ and $B$ so that the equation $x^2 - S(A)x + S(B) = 0$ has integer roots? *(N. Agakhanov, I. Bogdanov)*

7. In an acute-angled triangle $ABC$, $AA'$ and $BB'$ are altitudes. A point $D$ is chosen on the arc $ACB$ of the circumcircle of the triangle. Let the lines $AA'$ and $BD$ meet at $P$, and the lines $BB'$ and $AD$ meet at $Q$. Prove that the line $A'B'$ bisects the segment $PQ$. *(A. Akopyan)*

8. One hundred representatives of 50 countries, two from each country, are sitting at a round table. Prove that they can be partitioned into two gro-



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ups so that every group contains a representative from each country, and each person has at most one neighbor among the members of his group.

*(S. Berlov)*

**Grade 10**

*First Day*

1. Find the smallest natural number that is not representable in the form $\dfrac{2^a-2^b}{2^c-2^d}$, where $a,b,c,d$ are natural numbers. *(V. Senderov)*

2. Positive numbers are written in the cells of a $2\times n$ table so that in each of the $n$ columns the sum of two numbers is 1. Prove that one can erase one number from each column in such a way that the sum of the remaining numbers in either row is at most $\frac{n+1}{4}$. *(Ye. Kulikov)*

3. Distinct numbers are written on the reverse sides of 2005 cards (one number on each). In one step, it is allowed to choose three cards and ask for the set of numbers written on them. What is the smallest number of moves that is always sufficient to determine which number is written on each card? *(I. Bogdanov)*

4. Let $\omega_B$ and $\omega_C$ be the excircles of a triangle $ABC$ corresponding to $B$ and $C$ respectively. Circle $\omega_B'$ is symmetric to $\omega_B$ with respect to the midpoint of $AC$, and circle $\omega_C'$ is symmetric to $\omega_C$ with respect to the midpoint of $AB$. Prove that the line through the intersection points of $\omega_B'$ and $\omega_C'$ bisects the perimeter of $\triangle ABC$. *(P. Kozhevnikov)*

*Second Day*

5. Sixteen rooks are placed on the cells of an $8\times 8$ chessboard. What is the minimnum number of pairs of rooks attacking each other? *(Ye. Kulikov)*

6. *Problem 7 for Grade 9.*

7. Suppose that positive integers $x,y$ satisfy the equation $2x^2-1=y^{15}$. Prove that if $x>1$ then $x$ is divisible by 5. *(V. Senderov)*

8. On an infinite sheet of paper with a square grid, a finite number of cells are colored black so that each black cell has an even number (0, 2 or 4) of neighboring white cells (two cells are neighboring is they share a side). Prove that every white cell can be colored red or green so that each black cell has an equal number of red and green neighboring cells.

*(A. Glebov, D. Von der Flaass)*

## Grade 11

*First Day*

1. Assuming that the number of solutions $N$ of the equation

$$|x-a_1|+\cdots+|x-a_{50}|=|x-b_1|+\cdots+|x-b_{50}|$$

is finite, where $a_i, b_j$ are distinct numbers, what is the greatest possible value of $N$? *(I. Rubanov)*

2. *Problem 3 for Grade 10.*

3. The excircles of a triangle $ABC$ touch the corresponding sides $BC, CA$, $AB$ at $A', B', C'$, respectively. The circumcircles of the triangles $A'B'C$, $AB'C'$, and $A'BC'$ meet the circumcircle of $\triangle ABC$ again at $C_1, A_1, B_1$, respectively. Prove that the triangle $A_1B_1C_1$ is similar to the triangle whose vertices are at the points of tangency of the incircle of $\triangle ABC$ with its sides. *(L. Emelyanov)*

4. Positive integers $x, y, z$ with $x > 2$ and $y > 1$ satisfy $x^y + 1 = z^2$. Denote by $p$ the number of distinct prime divisors of $x$ and by $q$ that of $y$. Show that $p \geq q+2$. *(V. Senderov)*

*Second Day*

5. Does there exist a bounded function $f : \mathbb{R} \to \mathbb{R}$ with $f(1) > 0$ satisfying

$$f(x+y)^2 \geq f(x)^2 + 2f(xy) + f(y)^2?$$

*(N. Agakhanov)*

6. Can one place 12 rectangular parallelepipeds $P_1, P_2, \ldots, P_{12}$ in space, with sides parallel to the coordinate axes $Ox, Oy, Oz$, in such a way that for each $i = 1, \ldots, 12$, $P_i$ intersects (i.e. has a common point with) each of the remaining parallelepipeds except for $P_{i-1}$ and $P_{i+1}$? (Here $P_0 = P_{12}$ and $P_{13} = P_1$.) *(A. Akopyan)*

7. A circle with center $O$ is inscribed in a convex quadrilateral $ABCD$ with no two parallel sides. Prove that $O$ coincides with the intersection point of the two lines joining the midpoints of the opposite sides of $ABCD$ if and only if $OA \cdot OC = OB \cdot OD$. *(A. Zaslavskiy, M. Isayev, D. Tsvetov)*

8. One hundred representatives of 25 countries, four from each country, are sitting at a round table. Prove that they can be partitioned into four groups so that every group contains a representative of each country, and no two persons from the same group are sitting next to each other.

*(S. Berlov)*