35-th All-Russian Mathematical Olympiad, Kislovodsk, April, 21–27, 2009

Final Round

Grade 9

First Day

1. The denominators of two irreducible fractions are 600 and 700. Find the minimum value of the denominator of their sum (written as irreducible fraction).

2. Let BD ($B \in BC$) be the angle bisector in $\triangle ABC$. The line BD intersects the circumcircle Ω of $\triangle ABC$ at B and E. Circle ω with diameter DE intersects Ω again at F. Prove that BF is a symmedian of $\triangle ABC$.

3. Let a and n be positive integers such that $1 < n^2 < a$. If among the numbers $a+1, a+2, \dots, a+n$ one can find a multiple of each of the numbers $n^2+1, n^2+2, \dots, n^2+n$, prove that $a > n^4 - n^3$.

4. n cups are arranged on a circle. A coin is hidden under one of the cups. Each move consists of choosing 4 of the cups and examining whether the coin lies under one of them. After that, the cups are turned over and the coin moves to one of two places neighboring the original position of the coin. What is the minimal number of moves that guarantees that the coin will be found?

Second Day

5. Let a,bc be three real numbers that satisfy:

$$\begin{cases} (a+b)(b+c)(c+a) &= abc \\ (a^3+b^3)(b^3+c^3)(c^3+a^3) &= a^3b^3c^3. \end{cases}$$

   Prove that $abc = 0$.

6. Is it possible to color the positive integers in 2009 colors satisfying the following conditions:

   (i) Each color is used infinitely many times;

   (ii) There are no three numbers of the same color such that one of them is equal to the product of other two?

7. Eight squares on one of the diagonals of a chessboard are called *fence*. A rook moves on a board and never visits the same square twice (squares over which rook passes are not considered to be visited). The rook doesn't visit the squares of the *fence*. What is the maximal number of times the rook can jump over the *fence*?

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8. Triangles ABC and $A_1B_1C_1$ have the same area. Using a compass and a ruler, is it always possible to construct a triangle $A_2B_2C_2$ such that $\triangle A_2B_2C_2 \cong \triangle ABC$ and $AA_2 \| BB_2 \| CC_2$?

Grade 10

First Day

1. Find all values for n for which there are nonzero real numbers a, b, c, d such that the expansion of the polynomial $(ax+b)^{100} - (cx+d)^{100}$ has exactly n nonzero coefficients.

2. *Problem 2 for Grade 9.*

3. How many times does the function
$$f(x) = \cos x \cos \frac{x}{2} \cos \frac{x}{3} \cdots \cos \frac{x}{2009}$$
change its sign on the interval $[0, 2009\pi/2]$?

4. There are 2009 nonnegative integers ≤ 100 on a circle. Two numbers that are immediate neighbors of each other can be simultaneously increased by 1. This operation can be repeated at most k times. What is the minimal k that can make all the numbers of the circle equal regardless of their initial distribution?

Second Day

5. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of positive integers such that each term is divisible by either 1005 or 1006, but none of the terms is divisible by 97. Find the least possible value of the maximal difference of consecutive terms.

6. Given a finite tree T and an isomorphism $f : T \to T$, assume that $f(a) \neq a$ for every vertex a. Prove that there are vertices a and b such that $f(a) = b$ and $f(b) = a$.

7. Let I be the center of the incircle of $\triangle ABC$. Assume that the incircle touches the sides BC, CA, and AB at A_1, B_1, C_1 respectively. Denote by ω_B and ω_C the incircles of the quadrilaterals BA_1IC_1 and CA_1IB_1. Prove that the internal common tangent of ω_B and ω_C different from IA_1 passes through A.

8. Let x and y be two integers such that $2 \leq x, y \leq 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.

Grade 11

First Day

1. Some cities in a country are linked by roads. The roads meet each other in the cities only. Each city contains a board showing the shortest length of a trip (chain of roads) originating in that city and passing through all other cities (this shortest trip may pass through some of the cities more than once and doesn't have to end in the original city). Prove that for each two numbers a and b written on these boards we have $a \leq 1.5b$ and $b \leq 1.5a$.

2. Consider the sequence of numbers (a_n) $(n = 1,2,\dots)$ defined as follows: $a_1 \in (1,2)$, $a_{k+1} = a_k + \frac{k}{a_k}$ $(k = 1,2,\dots)$. Prove that there is at most two terms of this sequence whose sum is an integer.

3. Let $ABCD$ be a triangular pyramid such that no face of the pyramid is a right triangle and the orthocenters of triangles ABC, ABD, and ACD are collinear. Prove that the center of the sphere circumscribed about the pyramid belongs to the plane passing through the midpoints of AB, AC, and AD.

4. Let M be the set of points (x,y) in the plane such that $x,y \in \mathbb{Z}$ and $x^2 + y^2 \leq 10^{10}$. Two players play the following game: First player puts a coin on some of the points in M. After that the players move the coin such that in each of the moves a player puts the coin to some other point of M and records the distance that the coin has traveled. This distance must strictly increase in each move and a player is never allowed to put a coin in the place that is symmetric to its current position with respect to the origin. The winner is the player who makes the last move. Which player has a winning strategy?

Second Day

5. Prove that
$$\log_a b + \log_b c + \log_c a \leq \log_b a + \log_c b + \log_a c$$
for all $1 < a \leq b \leq c$.

6. There are k rooks on a 10×10 board. All the squares that at least one rook can capture are marked (squares where the rooks stand are captured). What is the maximum value of k for which the rooks can be arranged such that: after removing any rook from the chessboard, there is at least one marked square not captured by any of the remaining rooks?

7. Two points A_1 and C_1 are given on the sides AB and BC of a parallelogram $ABCD$. Let P be the intersection of the lines AC_1 and CA_1. Assume that the circumcircles of $\triangle AA_1P$ and $\triangle CC_1P$ intersect for the second time at the point Q inside $\triangle ACD$. Prove that $\angle PDA = \angle QBA$.

8. Let x and y be two integers such that $2 \leq x,y \leq 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.

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