29-th All-Russian Mathematical Olympiad 2003

Fourth Round

Grade 8

First Day

1. The numbers 1 through 10 are to be divided into two groups so that the product of numbers in the first group is divisible by that in the second. What is the smallest possible value of the quotient upon this division?

2. On each of two given intersecting lines in the coordinate plane there is a fly flying in a fixed direction with a constant speed. It is known that the projections of the flies on axis Ox never coincide. Show that their projections on axis Oy must coincide at some (past or future) moment.

3. Two players alternate writing natural numbers not exceeding 1000 on the board. The first player writes down number 1. Thereafter, number a on the board can be replaced by either $a+1$ or $2a$ (as long as it does not exceed 1000). The player who writes number 1000 loses the game. Who has a winning strategy?

4. Prove that every triangle can be cut into three poligons, one being an obtuse-angled triangle, which can be reassembled into a rectangle (with possible overlappings).

Second Day

5. At the vertices of a cube the numbers from 1 to 8 are written, and on each edge the absolute difference of the numbers written at its endpoints. Among the numbers on the edges, at least how many of them can be different?

6. Natural numbers a, b, c, d satisfy $\frac{a}{c} = \frac{b}{d} = \frac{ab+1}{cd+1}$. Prove that $a = c$ and $b = d$.

7. The triangle ABC has a right angle at C. Point D is taken on side AC and point K on segment BD so that $\angle ABC = \angle KAD = \angle AKD$. Show that $BK = 2DC$.

8. A set of 2003 positive numbers has the property that, whenever a and b are in the set ($a > b$), so is at least one of the numbers $a+b$ and $a-b$. Prove that if the numbers from the set are arranged in increasing order, then the differences between successive numbers will be equal.

Grade 9

First Day

1. Show that the sides of every scalene triangle can be either all lengthened or all shortened by the same segment so that the obtained segments are sides of a right-angled triangle.

2. *Problem 2 for Grade 8.*

3. In a triangle ABC with $AB = BC$ the middle line parallel to side BC meets the incircle at the midpoint of AC and at another point F. Prove that the tangent to the incircle at F and the bisector of angle C meet on the side AB.

4. Two players alternate writing on the board arbitrary (decimal) digits from left to right. The loser is the one after whose move a number formed by several successive digits (in that order) is divisible by 11. Which player has a winning strategy?

Second Day

5. Let I be the incenter of triangle ABC. Points A', B', C' are symmetric to I with respect to the sides of the triangle. Prove that if the circumcircle of triangle $A'B'C'$ passes through B, then $\angle ABC = 60^\circ$.

6. A hundred people were invited for a dinner. The guests who knew noone of the other guests entered first. Then those who knew exactly 1 of the remaining guests entered, then those knowing exactly 2 remaining guests, etc. to 99. At most how many people could have remaining outside at the end of this procedure?

7. Prove that from any six four-digit numbers whose greatest common divisor is 1 one can choose five whose greatest common divisor is also 1.

8. Prove that a convex polygon can be dissected by nonintersecting diagonals into acute-angled triangles in at most one way.

Grade 10

First Day

1. Find all angles α for which the set $\{\sin\alpha, \sin 2\alpha, \sin 3\alpha\}$ coincides with the set $\{\cos\alpha, \cos 2\alpha, \cos 3\alpha\}$.

2. *Problem 3 for Grade 9.*

3. There were 45 persons at a meeting. It turned out that any two of them who have a common acquaintance among the present persons do not know each other. What is the greatest possible number of pairs of people on the meeting who know each other?

4. In the plane are selected $n > 2$ lines passing through a point O with the property that, for any two of them, there is a selected line that bisects one of the two angles formed by the two lines. Show that the selected lines divide the full angle of 360° into n equal parts.

Second Day

5. For which x does the equation $x^2 + y^2 + z^2 + 2xyz = 1$ have a real solution for every y?

6. In a triangle ABC, A_0 is the midpoint of BC and A' is the tangency point of the incircle with BC. We draw the circle ω centered at A_0 and passing through A' and construct the analogous circles on the other two sides of triangle ABC. Prove that if ω touches the circumcircle of $\triangle ABC$ on the arc BC not containing A, then at least one of the other two constructed circles also touches the circumcircle.

7. Prove that from any set of at least four three-digit numbers whose greatest common divisor is 1 one can choose four whose greatest common divisor is also 1.

8. Two of the 17 apparently equal coins are fake and differ from the true ones in weight. It is known that the total weight of the two fakes equals twice the weight of a true coin. Is it always possible to find both fakes with five measurings on a ballance without weights?

Grade 11

First Day

1. Find all primes p for which there exist natural numbers x and y satisfying $p^x = y^3 + 1$.

2. Point K on the diagonal AC of a convex quadrilateral $ABCD$ is such that $KD = DC$, $\angle BAC = \frac{1}{2}\angle KDC$, and $\angle DAC = \frac{1}{2}\angle KBC$. Prove that $\angle KDA = \angle BCA$ or $\angle KDA = \angle KBA$.

3. The functions $f(x) - x$ and $f(x^2) - x^6$ are defined and increasing for positive x. Prove that the function $f(x^3) - \frac{\sqrt{3}}{2}x^6$ is also increasing for positive x.

4. Points $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$ are given on the plane. Show that the points B_i can be renumerated in such a way that the angle between $\overrightarrow{A_iA_j}$ and $\overrightarrow{B_iB_j}$ is acute or right whenever $i \neq j$.

Second Day

5. Quadratic trinomials $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + cx + d$ have the property that the equation $P(Q(x)) = Q(P(x))$ has no real roots. Prove that $b \neq d$.

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6. *Problem 6 for Grade 9.*

7. The sphere inscribed in tetrahedron $ABCD$ touches the face ABC at point T. A sphere ω' touches the face ABC at T' and also touches the exhensions of the other three faces. Show that the lines AT and AT' are symmetric with respect to the bisector of angle BAC.

8. *Problem 8 for Grade 10.*