17-th All-Russian Mathematical Olympiad 1991

Final (Fourth) Round – March 22–29

Grade 9

First Day

1. Find the locus of the foci of the parabolas given by $y = -x^2 + bx + c$ which touch the parabola $y = x^2$.

2. Let C be a point on the diameter AB of a semicircle with center O, distinct from A, B, O. Two perpendicular rays through C intersect the semicircle at points D and E. The line through D perpendicular to DC meets the semicircle again at K. Prove that if $K \neq E$, then KE and AB are parallel.

3. Three distinct positive numbers are given. Show that one can label them by a, b, c in such a way that
$$\frac{a}{b} + \frac{b}{c} > \frac{a}{c} + \frac{c}{a}.$$

4. A piece stands on the leftmost cell of a 1×100 board divided into unit squares. Two players alternately move the piece by 1, 10 or 11 cells to the right. The player who cannot perform a move loses the game. Which player can force a victory?

Second Day

5. Do there exist two integers whose sum of cubes equals 1991?

6. Points K and M are taken on the diagonal AC and points P and T on the diagonal BD of a convex quadrilateral $ABCD$, so that $AK = MC = \frac{1}{4}AC$ and $BP = TD = \frac{1}{4}BD$. Prove that the line passing through the midpoints of AD and BC bisects the segments PM and KT.

7. A wooden $n \times n$ board is divided into unit squares by lines parallel to the sides. Two players alternately make cuts of length 1 along these lines, starting from the border or from a point at a previously made cut. The player after whose move the board breaks loses the game. Which player can win no matter how the other player plays?

8. Each of the numbers $a_1, a_2, \ldots, a_n$ is greater than 1 and $|a_{k+1} - a_k| < 1$ for $k = 1, \ldots, n-1$. Prove that
$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

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Grade 10

First Day

1. Find all natural numbers p and q for which the equation $x^2 - pqx + p + q = 0$ has integral roots.

2. The segments connecting a point K to the vertices A and D of a rectangle $ABCD$ intersect the side BC. The perpendiculars from B and C to DK and AK, respectively, intersect at M. Show that if $M \neq K$, then MK is perpendicular to AD.

3. A polygon-shaped city is divided into areas by straight streets. At each vertex of the polygon there is a city square. Each street connects two squares and passes through no other square. Each street is one-way and it is possible to (i) enter every square; (ii) leave every square; (iii) go round the city along its boundary. Show that it is possible to go round at least one of the city areas.

4. A board with 6 columns and $n \geq 2$ rows is filled with zeros and ones in such a way that all the rows are different and, for every two rows $(a_1, \dots, a_6)$ and $(b_1, \dots, b_6)$, there is a row $(a_1 b_1, \dots, a_6 b_6)$. Show that in each column at least half the entries are zeros.

Second Day

5. At each vertex of a cube there is a fly. At one moment, each fly moves to another vertex, one fly to each vertex. Show that there exist three flies which form a triangle congruent to the one they formed initially.

6. An 11×12 rectangle is given. Show that

   (a) the rectangle can be tiled with 20 rectangles of sizes 1×6 or 1×7;

   (b) it cannot be tiled with 19 such rectangles.

7. Solve the system

$$5x\left(1 + \frac{1}{x^2 + y^2}\right) = 12, \quad 5y\left(1 - \frac{1}{x^2 + y^2}\right) = 4.$$

8. Delegates elect a committee as follows. Each delegate votes for 10 persons among the candidates. A committee is said to be *good* for a delegate if he voted for at least one of its members. Suppose that for any six delegates there is a two-member committee which is good for all the six delegates. Show that one can elect a 10-member committee which is good for all delegates.

Grade 11

First Day

1. The polynomial $2x^3 - 60x^2 + ax$ takes three consecutive integer values at three consecutive integer points (in the same order). Find these integer values.

2. The altitudes AD, BE, CF of an acute triangle ABC meet at H. Suppose that the areas of the quadrilaterals $AEHF$ and $HECD$ are equal. Show that the triangle ABC is isosceles.

3. The sum of positive numbers a, b, c is 1. Prove that

$$(1+a)(1+b)(1+c) \geq 8(1-a)(1-b)(1-c).$$

4. A cube of side n consists of n^3 unit cubes, where n is even. One arbitrarily selects $3n^2/2$ unit cubes. Prove that there is a right-angled triangle whose vertices lie in the centers of three of the selected unit cubes and whose legs are parallel to edges of the cube.

Second Day

5. Prove that for all x, y the following inequality holds:

$$\cos x + \cos y + 2\cos(x+y) \geq -\frac{9}{4}.$$

6. Circles S_1 and S_2 intersect at points A_1 and A_4, circles S_2 and S_3 at A_2, A_5, and circles S_3 and S_1 at A_3, A_6. A polygonal line $M_1 M_2 \dots M_7$ is such that each line $M_k M_{k+1}$ contains point A_k and points M_k, M_{k+1} lie on the circles which meet at A_k. Prove that the points M_1 and M_7 coincide.

7. Three lines a, b, c are given in space. Points $T_0, T_1, T_2, T_3, T_4, T_5, T_6$ are taken on lines a, b, c, a, b, c, a, respectively, so that $T_0T_1 \perp b$, $T_1T_2 \perp c$, $T_2T_3 \perp a$, $T_3T_4 \perp b$, $T_4T_5 \perp c$, $T_5T_6 \perp a$. Prove that if T_0 and T_6 coincide, then so do T_0 and T_3.

8. We are given $n^2 + n$ *corners*, each consisting of two perpendicular iron bars of length 1 with a common endpoint. These corners are used to form a square grid consisting of n^2 unit cells. Show that the number of corners with the legs showing up and right is equal to the number of corners with the legs showing down and left.

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