19-th All-Russian Mathematical Olympiad 1993

Fourth Round

Grade 9

First Day

1. If a and b are positive numbers, prove the inequality
$$a^2+ab+b^2 \geq 3(a+b-1).$$

2. Find the largest natural number which cannot be turned into a multiple of 11 by reordering its (decimal) digits.

3. Points M and N are chosen on the sides AB and BC of a triangle ABC. The segments AN and CM meet at O such that $AO = CO$. Is the triangle ABC necessarily isosceles, if
   (a) $AM = CN$?
   (b) $BM = BN$?

4. We have a deck of n playing cards, some of which are turned up and some are turned down. In each step we are allowed to take a set of several cards from the top, turn the set and place it back on the top of the deck. What is the smallest number of steps necessary to make all cards in the deck turned down, independent of the initial configuration?

Second Day

5. Show that the equation $x^3+y^3 = 4(x^2y+xy^2+1)$ has no integer solutions.

6. Three right-angled triangles have been placed in a halfplane determined by a line l, each with one leg lying on l. Assume that there is a line parallel to l cutting the triangles in three congruent segments. Show that, if each of the triangles is rotated so that its other leg lies on l, then there still exists a line parallel to l cutting them in three congruent segments.

7. Let E be an arbitrary point on the diagonal AC of a rhombus $ABCD$, distinct from A and C, and N,M be points on the lines AB,BC respectively, distinct from A and C, such that $AE = NE$ and $CE = ME$. Lines AM and CN intersect in point K. Show that K,E and D are collinear.

8. Number 0 is written on the board. Two players alternate writing signs and numbers to the right, where the first player always writes either $+$ or $-$ sign, while the second player writes one of the numbers $1,2,\dots,1993$, writing each of these numbers exactly once. The game ends after 1993 moves. Then the second player wins the score equal to the absolute value of the expression obtained thereby on the board. What largest score can he always win?

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Grade 10

First Day

1. Point D is chosen on the side AC of an acute-angled triangle ABC. The median AM intersects the altitude CH and the segment BD at points N and K respectively. Prove that if $AK = BK$, then $AN = 2KM$.

2. *Problem 2 for Grade 9.*

3. Solve in positive numbers the system

$$x_1 + \frac{1}{x_2} = 4, \quad x_2 + \frac{1}{x_3} = 1, \quad \cdots \quad x_{99} + \frac{1}{x_{100}} = 4, \quad x_{100} + \frac{1}{x_1} = 1.$$

4. Each citizen of town N knows at least 30% of the remaining citizens. A citizen votes in elections if he/she knows at least one candidate. Prove that it is possible to schedule elections with two candidates for the mayor of city N so that at least half the citizens of N can vote.

Second Day

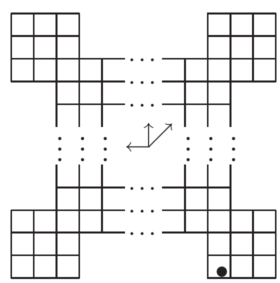
5. *Problem 5 for Grade 9.*

6. Prove that

$$\sqrt{2+\sqrt[3]{3+\cdots+\sqrt[1993]{1993}}} < 2.$$

7. Points M and N are taken on the sides BC and CD respectively of a parallelogram $ABCD$. Diagonal BD meets AM at E and AN at F, thus cutting triangle AMN into two parts. Prove that these two parts have equal areas if and only if the point K given by $EK \parallel AD$ and $FK \parallel AB$ lies on the segment MN.

8. From a square board 1000×1000 four rectangles 2×994 have been cut off as shown on the picture. Initially, on the marked square there is a *centaur* - a piece that moves to the adjacent square to the left, up, or diagonally up-right in each move. Two players alternately move the centaur. The one who cannot make a move loses the game. Who has a winning strategy?



Grade 11

First Day

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1. Find all natural numbers n for which the sum of digits of 5^n equals 2^n.

2. Prove that, for every integer $n > 2$, the number $\left[\left(\sqrt[3]{n}+\sqrt[3]{n+2}\right)^3\right]+1$ is divisible by 8.

3. Point O is the foot of the altitude of a quadrilateral pyramid. A sphere with center O is tangent to all lateral faces of the pyramid. Points A, B, C, D are taken on successive lateral edges so that segments AB, BC, and CD pass through the three corresponding tangency points of the sphere with the faces. Prove that the segment AD passes through the fourth tangency point.

4. Given a regular $2n$-gon, show that each of its sides and diagonals can be assigned an arrow in such a way that the sum of the obtained vectors equals zero.

Second Day

5. The expression $x^3 + \dots x^2 + \dots x + \dots = 0$ is written on the blackboard. Two pupils alternately replace the dots by real numbers. The first pupil attempts to obtain an equation having exactly one real root. Can his opponent spoil his efforts?

6. Seven tetrahedra are placed on the table. For any three of them there exists a horizontal plane cutting them in triangles of equal areas. Show that there exists a plane cutting all seven tetrahedra in triangles of equal areas.

7. Let ABC be an equilateral triangle. For an arbitrary line l through B, the orthogonal projections of A and C on l are denoted by D and E respectively. If $D \neq E$, equilateral triangles DEP and DET are constructed on different sides of l. Find the loci of P and T.

8. There are 1993 towns in a country, and at least 93 roads going out of each town (to another town). It is known that every town can be reached from every other town by roads. Prove that this can always be done with not more than 62 transfers.