19-th All-Russian Mathematical Olympiad 1993

Final Round – Anapa, April

Grade 9

First Day

1. A natural number n is such that $2n+1$ and $3n+1$ are perfect squares. Can the number $5n+3$ be prime? *(Ye. Gladkova)*

2. The segments AB and CD of the unit length intersect at a point O, where $\angle AOC = 60^\circ$. Prove that $AC + BD \geq 1$. *(S. Berlov)*

3. A quadratic polynomial $f(x)$ can be replaced either by $x^2 f\left(1+\frac{1}{x}\right)$, or by $(x-1)^2 f\left(\frac{1}{x-1}\right)$. Starting with the polynomial x^2+4x+3, is it posible to obtain the polynomial $x^2+10x+9$? *(A. Perlin)*

4. In a photo-album there are (a) 10, (b) n photos. On each photo there are three persons: a male in the middle, his son to the left, and his brother to the right. What is the smallest possible number of different people on these photos, assuming that all the men in the middle of the photos are different? *(S. Konyagin)*

Second Day

5. Positive integers x, y, z satisfy the equality $(x-y)(y-z)(z-x) = x+y+z$. Prove that $x+y+z$ is divisible by 27. *(N. Agakhanov)*

6. A convex quadrilateral is placed inside a circle, and the extensions of its sides intersect the circle at points $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$, where the points appear in this order on the circle. Show that if $A_1B_2 = B_1C_2 = C_1D_2 = D_1A_2$, then the quadrilateral formed by the lines $A_1A_2, B_1B_2, C_1C_2, D_1D_2$ is cyclic. *(D. Tereshin)*

7. What is the greatest number of pawns that can be placed on cells of a chessboard so that on every row, column or diagonal there is an even number of pawns? *(S. Zaptsev)*

8. An odd number n of expressions of the form $*x^2 + *x + * = 0$ are written on a blackboard. Two players alternately replace the stars in these expressions with nonzero real numbers. After $3n$ moves n quadratic equations are obtained. The first player strives to maximize the number of equations having no real roots, while the second tries to spoil his efforts. What is the maximum number of equations with no real roots the first player can obtain independently of how well the second player plays? *(I. Rubanov)*

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Grade 10

First Day

1. The side lengths of a triangle are prime numbers. Prove that its area cannot be an integer. *(D. Mit'kin)*

2. Two lines are drawn through the center of symmetry of two intersecting congruent circles. The two lines intersect the circles at four non-collinear points. Show that these points lie on a circle. *(L. Kuptsov)*

3. *Problem 3 for Grade 9.*

4. Thirty persons from a company sit at a round table. Some of them are clever man and some are jackasses. Each person is asked the question: "Is your neighbor to the right a clever man or a jackass?". A clever man answers correctly, while a jackass can answer either correctly or incorrectly. Assume that there are no more than F jackasses. What is the maximum integer value of F for which it is always possible, knowing the answers, to identify a clever man in the company? *(O. Lyashko)*

Second Day

5. *Problem 5 for Grade 9.*

6. Is it true that every two rectangles of the same area can be placed on the coordinate plane in such a way, that every horizontal line intersecting one of them intersects the other as well, and the determined segments have the same length? *(D. Tereshin)*

7. A square with side n is divided into n^2 unit squares. What is the largest n for which one can mark n unit squares such that every rectangle, having sides at gridlines and area not exceeding n, contains at least one marked square in its interior? *(D. Van der Flaass)*

8. For any sequence (a_k) of real numbers, define its *median sequence* (a'_k) by $a'_k = \dfrac{a_k + a_{k+1}}{2}$. Consider the sequences (a_k), (a'_k), (a''_k) etc, where every sequence is the median of its precedent. If the terms of all these sequences are integers, we say that the sequence (a_k) is *good*. Prove that if (a_k) is good, then so is the sequence (a_k^2). *(D. Tamarkin)*

Grade 11

First Day

1. *Problem 1 for Grade 9.*

2. Two right-angled triangles are situated on a plane so that their medians from the right angles are parallel. Prove that the angle between some leg of one triangle and some leg of the other is half the angle between their hypotenuses. *(K. Fel'dman)*

3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition
$$f(x^y) = f(x)^{f(y)} \quad \text{for all } x, y > 0.$$ *(S. Tokarev)*

4. Show that there exists a natural number n with the following property: If a regular triangle with side n is divided into n^2 regular triangles with side 1, then among the vertices of these triangles one can choose $1993n$ points, no three of which are vertices of a regular triangle.
(S. Avgustinovich, D. Van der Flaass)

Second Day

5. Find all quadruples of real numbers with the property that in each quadruple, each term equals the product of some two other terms. *(D. Mit'kin)*

6. The numbers 1 through 1993 in some order are written in a row. The following operation is performed: if the first number in the row is k, then the first k numbers are reversed. Show that after several such operations number 1 will occur on the first place. *(D. Tereshin)*

7. At a tennis tournament with n participants, matches were played in pairs (two against two) so that any two participants played exactly one match against each other. For which n is this possible? *(S. Tokarev)*

8. Prove that any two rectangular parallelepipeds of the same volume can be placed in space in such a way, that every horizontal plane intersecting one of them intersects the other as well, and the polygons at the intersection have the same area?
(D. Tereshin)