21-st All-Russian Mathematical Olympiad 1995

Final Round – Saratov

Grade 9

First Day

1. A freight train departed from Moscow at x hours and y minutes and arrived at Saratov at y hours and z minutes. The length of its trip was z hours and x minutes. Find all possible values of x. *(S. Tokarev)*

2. A chord CD of a circle with center O is perpendicular to a diameter AB. A chord AE bisects the radius OC. Show that the line DE bisects the chord BC. *(V. Gordon)*

3. Can the equation $f(g(h(x))) = 0$, where f,g,h are quadratic polynomials, have the solutions $1,2,3,4,5,6,7,8$? *(S. Tokarev)*

4. Can the numbers from 1 to 81 be written in a 9×9 board, so that the sum of numbers in each 3×3 square is the same? *(S. Tokarev)*

Second Day

5. We call natural numbers *similar* if they are written with the same (decimal) digits. For example, $112, 121, 211$ are similar numbers having the digits $1,1,2$. Show that there exist three similar 1995-digit numbers with no zero digits, such that the sum of two of them equals the third. *(S. Dvoryaninov)*

6. In an acute-angled triangle ABC, points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1, respectively. Compute the sum of angles $B_2A_1C_2$, $C_2B_1A_2$ and $A_2C_1B_2$. *(D. Tereshin)*

7. There are three boxes of stones. Sisyphus moves stones one by one between the boxes. Whenever he moves a stone, Zeus gives him the number of coins that is equal to the difference between the number of stones in the box the stone was put in, and that in the box the stone was taken from (the moved stone does not count). If this difference is negative, then Sisyphus returns the corresponding amount to Zeus (if Sisyphus cannot pay, generous Zeus allows him to make the move and pay later).

   After some time all the stones lie in their initial boxes. What is the greatest possible earning of Sisyphus at that moment? *(I. Izmest'ev)*

8. Numbers 1 and -1 are written in the cells of a board 2000×2000. It is known that the sum of all the numbers in the board is positive. Show that one can select 1000 rows and 1000 columns such that the sum of numbers written in their intersection cells is at least 1000. *(D. Karpov)*

Grade 10

First Day

1. Solve the equation $\cos\cos\cos\cos x = \sin\sin\sin\sin x$.
(V. Senderov, I. Yashchenko)

2. *Problem 2 for Grade 9.*

3. Does there exist a sequence of natural numbers in which every natural number occurs exactly once, such that for each $k = 1,2,3,\dots$ the sum of the first k terms of the sequence is divisible by k? *(A. Shapovalov)*

4. Prove that if all angles of a convex n-gon are equal, then there are at least two of its sides that are not longer than their adjacent sides.
(A. Berzin'sh, O. Musin)

Second Day

5. A sequence (a_i) of natural numbers has the property that for all $i \neq j$, $\gcd(a_i, a_j) = \gcd(i, j)$. Show that $a_i = i$ for all $i \in \mathbb{N}$. *(A. Golovanov)*

6. Let be given a semicircle with diameter AB and center O, and a line intersecting the semicircle at C and D and the line AB at M ($MB < MA$, $MD < MC$). The circumcircles of the triangles AOC and DOB meet again at L. Prove that $\angle MKO$ is right. *(L. Kuptsov)*

7. *Problem 8 for Grade 9.*

8. Let $P(x)$ and $Q(x)$ be monic polynomials. Prove that the sum of the squares of the coeficients of the polynomial $P(x)Q(x)$ is not smaller than the sum of the squares of the free coefficients of $P(x)$ and $Q(x)$.
(M. Min'ott)

Grade 11

First Day

1. Can the numbers $1,2,3,\dots,100$ be covered with 12 geometric progressions? *(A. Golovanov)*

2. Prove that every real function, defined on all of $\mathbb{R}$, can be represented as a sum of two functions whose graphs both have an axis of symmetry.
(D. Tereshin)

3. Two points on the distance 1 are given in a plane. It is allowed to draw a line through two marked points, as well as a circle centered in a marked point with radius equal to the distance between some two marked points.

By marked points we mean the two initial points and intersection points of two lines, two circles, or a line and a circle constructed so far. Let $C(n)$ be the minimum number of circles needed to construct two points on the distance n if only a compass is used, and let $LC(n)$ be the minimum total number of circles and lines needed to do so if a ruler and a compass are used, where n is a natural number. Prove that the sequence $C(n)/LC(n)$ is not bounded. *(A. Belov)*

4. *Problem 4 for Grade 10.*

Second Day

5. Prove that for every natural number a_1 there exists an increasing sequence of natural numbers (a_n) such that $a_1^2+a_2^2+\cdots+a_k^2$ is divisible by $a_1+a_2+\cdots+a_k$ for all $k \geq 1$. *(A. Golovanov)*

6. A boy goes at a merry-go-round with n seats n times. After every time he moves in the clockwise direction and takes another seat, not making a full circle. The number of seats he passes by at each move is called the length of the move. For which n can he sit at every seat, if the lengths of all the $n-1$ moves he makes have different lengths? *(V. New)*

7. The altitudes of a tetrahedron intersect in a point. Prove that this point, the foot of one of the altitudes, and the points dividing the other three altitudes in the ratio 2 : 1 (measuring from the vertices) lie on a sphere *(D. Tereshin)*

8. *Problem 8 for Grade 10.*

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com