46-th Federal Mathematical Competition of Serbia and Montenegro 2006

High School

Vršac, April 15, 2006

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1-st Grade

1. In a convex quadrilateral $ABCD$, $\angle BAC = \angle DAC = 55^\circ$, $\angle DCA = 20^\circ$, and $\angle BCA = 15^\circ$. Find the measure of $\angle DBA$.

2. Let x, y, z be positive numbers with $x + y + z = 1$. Show that

$$yz + zx + xy \geq 4(y^2z^2 + z^2x^2 + x^2y^2) + 5xyz.$$

   When does equality hold?

3. Determine the largest natural number whose all decimal digits are different and which is divisible by each of its digits.

4. Tatjana imagined a polynomial $P(x)$ with nonnegative integer coefficients. Danica is trying to guess the polynomial. In each step, she chooses an integer k and Tatjana tells her the value of $P(k)$. Find the smallest number of steps Danica needs in order to find the polynomial Tatjana imagined.

2-nd Grade

1. Suppose a, b, c, A, B, C are real numbers with $a \neq 0$ and $A \neq 0$ such that for all x,

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|.$$

   Prove that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

2. For an arbitrary point M inside a given square $ABCD$, let T_1, T_2, T_3 be the centroids of triangles ABM, BCM, and DAM, respectively. Let O_M be the circumcenter of triangle $T_1T_2T_3$. Find the locus of points O_M when M takes all positions within the interior of the square.

3. For every natural number a, consider the set $S(a) = \{a^n + a + 1 \mid n = 2, 3, \dots\}$. Does there exist an infinite set $A \subset \mathbb{N}$ with the property that for any two distinct elements $x, y \in A$, x and y are coprime and $S(x) \cap S(y) = \emptyset$?

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4. There are n coins aligned in a row. In each step, it is allowed to choose a coin with the tail up (but not one of outermost markers), remove it and reverse the closest coin to the left and the closest coin to the right of it. Initially, all the coins have tails up. Prove that one can achieve the state with only two coins remaining if and only if $n-1$ is not divisible by 3.

3-rd and 4-th Grades

1. Let x, y, z be positive numbers with the sum 1. Prove that
$$\frac{x}{y^2+z}+\frac{y}{z^2+x}+\frac{z}{x^2+y} \geq \frac{9}{4}.$$
When does equality hold?

2. Given prime numbers p and q with $p<q$, determine all pairs (x, y) of positive integers such that
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{p}-\frac{1}{q}.$$

3. Show that for an arbitrary tetrahedron there are two planes such that the ratio of the areas of the projections of the tetrahedron onto the two planes is not less than $\sqrt{2}$.

4. Miloš arranged the numbers 1 through 49 into the cells of a 7×7 board. Djordje wants to guess the arrangement of the numbers. He can choose a square covering some cells of the board and ask Miloš which numbers are found inside that square. At least, how many questions does Djordje need so as to be able to guess the arrangement of the numbers?

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