Chinese IMO Team Selection Test 2008

Time: 4.5 hours each day.

First Day

1. Let ABC be a triangle such that $AB > AC$. Let E be the point of tangency of BC with the incircle of ABC. Let D be the second intersection point of the incircle with the segment AE. Point $F \in AE$ $(F \neq E)$ satisfies $CE = CF$. The ray CF intersects BD at G. Prove that $CF = FG$.

2. The sequence (x_n) is defined by $x_1 = 2$, $x_2 = 12$, and $x_{n+2} = 6x_{n+1} - x_n$, for $n \geq 1$. Let p be an odd prime number and let q be a prime divisor of x_p. Prove that if $q \notin \{2,3\}$ then $q \geq 2p - 1$.

3. Suppose that every positive integer has been painted in one of the two colors. Prove that there exists an infinite sequence of positive integers $a_1 < a_2 < \cdots$ such that $a_1, \frac{a_1+a_2}{2}, a_2, \frac{a_2+a_3}{2}, a_3, \dots$ is an infinite sequence of positive integers of the same color.

Second Day

4. Let $n \geq 4$ be an integer. Consider all the subsets of $G_n = \{1,2,\dots,n\}$ with at least two elements. Prove it is possible to arrange those subsets in a sequence $P_1, P_2, \dots, P_{2^n-n-1}$ such that $|P_i \cap P_{i+1}| = 2$ for every $i = 1,2,\dots,2^n - n - 2$.

5. Let m, n be two positive integers. Positive real numbers $a_{i,j}$ $(1 \leq i \leq n, 1 \leq j \leq m)$ are not all equal to zero. If
$$f = \frac{n\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)^2 + m\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)^2}{\left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\right)^2 + mn\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2},$$
find the maximum and minimum of f.

6. Find the maximal constant M such that for arbitrary integer $n \geq 3$, there exists two sequences of positive real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that
   (i) $\sum_{k=1}^{n} b_k = 1$, $2b_k \geq b_{k-1} + b_{k+1}$, $k = 2,3,\dots,n-1$;
   (ii) $a_k^2 \leq 1 + \sum_{i=1}^{k} a_i b_i$, $k = 1,2,\dots,n$, $a_n = M$.