Bulgarian Mathematical Olympiad 2004
Regional Round, April 17-18

Grade 9

1. Find all values of a such that the equation
$$\sqrt{(4a^2-4a-1)x^2-2ax+1}=1-ax-x^2$$
has exactly two solutions. (Sava Grozdev, Svetlozar Doychev)

2. Let A_1 and B_1 be points on the sides AC and BC of $\triangle ABC$ such that $4\cdot AA_1\cdot BB_1=AB^2$. If $AC=BC$, prove that the line AB and the bisectors of $\angle AA_1B_1$ and $\angle BB_1A_1$ are concurrent. (Sava Grozdev, Svetlozar Doychev)

3. Let $a,b,c>0$ and $a+b+c=1$. Prove that
$$\frac{9}{10}\le\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}<1$$
(Sava Grozdev, Svetlozar Doychev)

4. Solve in integers the equation
$$x^3+10x-1=y^3+6y^2.$$
(Sava Grozdev, Svetlozar Doychev)

5. A square $n\times n$ $(n>2)$ is divided into n^2 unit squares colored in black or white such that the squares at the four corners of any rectangle (containing at least four squares) have no the same color. Find the maximum possible value of n. (Sava Grozdev, Svetlozar Doychev)

6. Consider the equations
$$[x]^3+x^2=x^3+[x]^2 \quad \text{and} \quad [x^3]+x^2=x^3+[x^2]$$
where $[t]$ is the greatest integer that does not exceed t. Prove that:
   (a) any solution of the first equation is an integer;
   (b) the second equation has a non-integer solution.

(Sava Grozdev, Svetlozar Doychev)

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Grade 10

1. Solve the inequality

$$\sqrt{x^2-1}+\sqrt{2x^2-3}+x\sqrt{3}>0$$

(Peter Boyvalenkov)

2. Let M be the centroid of $\triangle ABC$. Prove that:

   (a) $\cot\angle AMB=\dfrac{BC^2+CA^2-5AB^2}{12[ABC]}$;

   (b) $\cot\angle AMB+\cot BMC+\cot CMA\le -\sqrt{3}$.

(Peter Boyvalenkov)

3. In a school there are m boys and j girls, $m\ge 1$, $1\le j<2004$. Every student has sent a post card to every student. It is known that the number of the post cards sent by the boys is equal to the number of the post cards sent by girl to girl. Find all possible values of j. (Ivailo Kortezov)

4. Consider the function

$$f(x)=(a^2+4a+2)x^3+(a^3+4a^2+a+1)x^2+(2aa^2)x+a^2,$$

   where a is a real parameter.

   (a) Prove that $f(-a)=0$.

   (b) Find all values of a such that the equation $f(x)=0$ has three different positive roots.

(Ivan Landjev)

5. Let O and G be respectively the circumcenter and the centroid of $\triangle ABC$ and let M be the midpoint of the side AB. If $OG\perp CM$, prove that $\triangle ABC$ is isosceles. (Ivailo Kortezov)

6. Prove that any graph with 10 vertices and 26 edges contains least 4 triangles. (Ivan Landjev)

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Grade 11

1. Find all values of $x \in (-\pi, \pi)$ such that the numbers $2^{\sin x}$, $2 - 2^{\sin x + \cos x}$ and $2^{\cos x}$ are consecutive terms of a geometric progression. (Emil Kolev)

2. The lines through the vertices A and B that are tangent to circumcircle of an acute $\triangle ABC$ meet at a point D. If M is the midpoint the side AB, prove that $\angle ACM = \angle BCD$. (Emil Kolev)

3. Let $m \geq 3$ and $n \geq 2$ be integers. Prove that in a group of $N = mn - n + 1$ people such that there are two familiar people among any m, there is a person who is familiar with n people. Does the statement remain true if $N < mn - n + 1$? (Alexander Ivanov)

4. The points D and E lie respectively on the perpendicular biectors of the sides AB and BC of $\triangle ABC$. It is known that D is an interior point for $\triangle ABC$, E does not and $\angle ADB = \angle CEB$. If the line AE meets the segment CD at a point O, prove that the areas of $\triangle ACO$ and the quadrilateral $DBEO$ are equal. (Alexander Ivanov)

5. Let a, b and c be positive integers such that one of them is coprime with any of the other two. Prove that there are positive integers x, y and z such that $x^a = y^b + z^c$. (Alexander Ivanov)

6. One chooses a point in the interior of $\triangle ABC$ with area 1 and connects it with the vertices of the triangle. Then one chooses a point in the interior of one of the three new triangles and connects it with its vertices, etc. At any step one chooses a point in the interior of one of the triangles obtained before and connects it with the vertices of this triangle. Prove that after the n-th step:

   (a) $\triangle ABC$ is divided into $2n+1$ triangles;

   (b) there are two triangles with common side whose combined area is not less than $\frac{2}{2n+1}$.

   (Alexander Ivanov)

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Grade 12

1. Solve in integers the equation
$$2^a + 8b^2 - 3^c = 283.$$
(Oleg Mushkarov, Nikolai Nikolov)

2. Find all values of a such that the maximum of the function
$$f(x) = \frac{ax-1}{x^4 - x^2 + 1}$$
is equal to 1. (Oleg Mushkarov, Nikolai Nikolov)

3. A plane bisects the volume of the tetrahedron $ABCD$ and meets the edges AB and CD respectively at points M and N such that $\frac{AM}{BM} = \frac{CN}{DN} \neq 1$. Prove that the plane passes through the midpoints of the edges AC and BD. (Oleg Mushkarov, Nikolai Nikolov)

4. Let $ABCD$ be a circumscribed quadrilateral. Find $\angle BCD$ if $AC = BC$, $AD = 5$, $E = AC \cap BD$, $BE = 12$ and $DE = 3$. (Oleg Mushkarov, Nikolai Nikolov)

5. A set A of positive integers less than 2000000 is called *good* if $2000 \in A$ and a divides b for any $a, b \in A$, $a < b$. Find:
   (a) the maximum possible cardinality of a good set;
   (b) the number of the good sets of maximal cardinality.

   (Oleg Mushkarov, Nikolai Nikolov)

6. Find all non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients such that $P(x)Q(x+1) = P(x+2004)Q(x)$ for any x.
(Oleg Mushkarov, Nikolai Nikolov)

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