Croatian Team Selection Test 2009

1. Find all real numbers x, y, z such that the following two equations are satisfied

$$\begin{aligned} 3(x^2+y^2+z^2) &= 1, \\ x^2y^2+y^2z^2+z^2x^2 &= xyz(x+y+z)^3. \end{aligned}$$

2. Each positive integer is colored in one of the k colors. Prove that there exist four distinct natural numbers a, b, c, d, all of the same color, such that $ad = bc$, b/d is a power of 2 and c/a is a power of 3.

3. Let ABC be a triangle such that $AB > AC$. Let l be a tangent at A to the circumcircle of ABC. A circle with center A and radius AC intersects AB at D and the line l at E and F (in such a way that C and E are on the same side of AB). Prove that the line DE contains the incenter of ABC.

4. Determine all natural numbers n for which there exists natural number m divisible by all numbers from 1 to n but not divisible by any of the numbers $n+1$, $n+2$, $n+3$.