Bulgarian Mathematical Olympiad 2006
Regional Round, April 15-16

Grade 9

1. Find all real numbers a such that the roots x_1 and x_2 of the equation

$$x^2+6x+6a-a^2=0$$

satisfy the relation $x_2 = x_1^3 - 8x_1$. (Ivan Landjev)

2. Two circles k_1 and k_2 meet at points A and B. A line through B meets the circles k_1 and k_2 at points X and Y, respectively. The tangent lines to k_1 at X and to k_2 at Y meet at C. Prove that:

   (a) $\angle XAC = \angle BAY$.

   (b) $\angle XBA = \angle XBC$, if B is the midpoint of XY.

(Stoyan Atanasov)

3. The positive integers ℓ, m, n are such that $m-n$ is a prime number and

$$8\left(\ell^2 - mn\right) = 2(m2+n^2)+5(m+n)\ell$$

Prove that $11\ell+3$ is a perfect square. (Ivan Landjev)

4. Find all integers a such that the equation

$$x^4+2x^3+(a^2a9)x^2-4x+4=0$$

has at least one real root. (Stoyan Atanasov)

5. Given a right triangle ABC ($\angle ACB = 90^\circ$), let CH, $H \in AB$, be the altitude to AB and P and Q be the tangent points of the incircle of $\triangle ABC$ to AC and BC, respectively. If $AQ \perp HP$ find the ratio $\frac{AH}{BH}$.

(Stoyan Atanasov)

6. An air company operates 36 airlines in a country with 16 airports. Prove that one can make a round trip that includes 4 airports. (Ivan Landjev)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX and translation by Borislav Mirchev and Ercole Suppa
www.imomath.com

1. A circle k is tangent to the arms of an acute angle AOB at points A and B. Let AD be the diameter of k through A and $BP \perp AD$, $P \in AD$. The line OD meets BP at point M. Find the ratio $\frac{AH}{BH}$.

   (Peter Boyvaienkov)

2. Find the maximum of the function

$$f(x) = \frac{\log x \log x^2 + \log x^3 + 3}{\log^2 x + \log x^2 + 2}$$

   and the values of x, when it is attained. (Ivailo Kortezov)

3. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Find all funcns $f : \mathbb{Q}^+ \to \mathbb{R}$ such that $f(1) = 1$, $f(1/x) = f(x)$ for any $x \in \mathbb{Q}^+$ and $xf(x) = (x+1)f(x-1)$ for any $x \in \mathbb{Q}^+$, $x > 1$. (Ivailo Kortezov)

4. The price of a merchandize dropped from March to April by x%, and went up from April to May by y%. It turned out that in the period orn March to May the prize dropped by (y x)%. Find x and y if they are positive integers (the prize is positive for the whole period). (Ivailo Kortezov)

5. Let $ABCD$ be a parallelogram such that $\angle BAD < 90^\circ$ and DE, $E \in AB$, and DF, $F \in BC$, be the altitudes of the parallelogram. Prove that

$$4(AB \cdot BC \cdot EF + BD \cdot AE \cdot FC) \leq 5 \cdot AB \cdot BC \cdot BD.$$

   Find $\angle BAD$ if the equality occurs. (Ivailo Kortezov)

6. See problem 6 (grade 9).

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX and translation by Borislav Mirchev and Ercole Suppa
www.imomath.com

Grade 11

1. Let k be a circle with diameter AB and let $C \in k$ be an arbitrary point. The excircles of $\triangle ABC4$ tangent to the sides AC and BC are tangent to the line AB at points M and N, respectively. Denote by O_1 and O_2 the circumcenters of $\triangle AMC$ and $\triangle BNC$. Prove that the area of $\triangle O_1CO_2$ does not depend on C. (Alexander Ivanov)

2. Prove that

$$t^2(xy+yz+zx)+2t(x+y+z)+3 \geq 0 \quad \text{for all } x,y,z,t \in [1,1]$$

(Nikolai Nikolov)

3. Consider a set S of 2006 points in the plane. A pair $(A,B) \in SxS$ is called *isolated* if the disk with diameter AB does not contain other points from S. Find the maximum number of *isolated* pairs. (Alexander Ivanov)

4. Find the least positive integer a such that the system

$$\begin{cases} x+y+z=a \\ x^3+y^3+z^3=a \end{cases}$$

has no an integer solution. (Oleg Mushkarov)

5. The tangent lines to the circumcircle k of an isosceles $\triangle BAC$, $AC=BC$, at the points B and C meet at point X. If AX meets k at point Y, find the ratio AY/BY. (Emil Kola)

6. Let $a_1, a_2, \dots$ be a sequence of real numbers less than 1 and such that $a_{n+1}(a_n+2)=3$, $n \geq 1$. Prove that:

   (a) $-\frac{7}{2} < a_n < -2$;

   (b) $a_n = -3$ for any n.

(Nikolai Nikolov)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX and translation by Borislav Mirchev and Ercole Suppa
www.imomath.com

Grade 12

1. Find the area of the triangle determined by the straight line with equation $x - y + 1 = 0$ and the tangent lines to the graph of the parabola $y = x^2 - 4x + 5$ at its common points with the line. (Emil Kolev)

2. See problem n.5 (grade 11).

3. Find all real numbers a, such that the inequality

$$x^4 + 2ax^3 + a^2x^2 - 4x + 3 > 0$$

(Nikolai Nikolov)

4. Find all positive integers n for which the equality

$$\frac{\sin(n\alpha)}{\sin\alpha} - \frac{\cos(n\alpha)}{\cos\alpha} = n - 1$$

holds true for all $a \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}$. (Emil Kolev)

5. A plane intersects a tetrahedron $ABCD$ and divides the medians of the triangles DAB, DBC and DCA through D in ratios $1:2$, $1:3$ and $1:4$ from D, respectively. Find the ratio of the volumes of the two parts of the tetrahedron cut by the plane. (Oleg Mushkarov)

6. See problem 6 (grade 11).

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX and translation by Borislav Mirchev and Ercole Suppa
www.imomath.com