Bulgarian Mathematical Olympiad 2005
Regional Round, April 16-17

Grade 9

1. Find all values of the real parameters a and b such that the remainder in the division of the polynomial $x^4 - 3ax^3 + ax + b$ by the polynomial $x^2 1$ is equal to $(a^2 + 1)x + 3b^2$. (Peter Boyvalenkov)

2. Two tangent circles with centers O_1 and O_2 are inscribed in a given angle. Prove that if a third circle with center on the segment O_1O_2 is inscribed in the angle and passes through one of the points O_l and O_2 then it passes through the other one too. (Peter Boyvalenkov)

3. Let a and b be integers and k be a positive integer. Prove that if x and y are consecutive integers such that

$$a^k x b^k y = ab,$$

then $|ab|$ is a perfect k-th power. (Peter Boyvalenkov)

4. Find all values of the real parameter p such that the equation $| x^2 - px2p + l |= p1$ has four real roots x_1, x_2, x_3 and x_4 such that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 20.$$

(Ivailo Kortezov)

5. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The rays $\overrightarrow{DA}$ and $\overrightarrow{CB}$ meet at point N and the line NT is tangent to k, $T \in k$. The diagonals AC and BD meet at the centroid P of $\triangle NTD$. Find the ratio $NT : AP$. (Ivailo Kortezov)

6. A card game is played by five persons. In a group of 25 persons all like to play that game. Find the maximum possible number of games which can be played if no two players are allowed to play simultaneously more than once. (Ivailo Kortezov)

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Grade 10

1. Solve the system
$$\begin{cases} 3\cdot 4^x + 2^{x+1}\cdot 3^y 9^y = 0 \\ 2\cdot 4^x - 5\cdot 2^x 3^y + 9^y = -8 \end{cases}$$
(Ivan Landjev)

2. Given a quadrilateral $ABCD$ set $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$ and $BD = f$. Prove that:
   (a) $a^2 + b^2 + c^2 + d^2 \geq e^2 + f^2$;
   (b) if the quadrilateral $ABCD$ is cyclic then $|a - c| \geq |e - f|$.

   (Stoyan Atanassov)

3. Find all pairs of positive integers (m, n), $m > n$, such that
$$\left[m^2 + mn, mn - n^2\right] + [mn, mn] = 2^{2005}$$
where $[a, b]$ denotes the least common multiple of a and b.
(Ivan Landjev)

4. Find all values of the real parameter a such that the number of the solutions of the equation
$$3\left(5x^2a^4\right) - 2x = 2a^2(6x1)$$
does not exceed the number of the solutions of the equation
$$2x^3 + 6x = \left(3^{6a} - 9\right)\sqrt{2^{8a} - \frac{1}{6}} - (3a-1)^2 12^x$$
(Ivan Landjev)

5. Let H be the orthocenter of $\triangle ABC$, M be the midpoint of AB and H_1 and H_2 be the feet of the perpendiculars from H to the inner and the outer bisector of $\angle ACB$, respectively. Prove that the points H_1, H_2 and M are colinear. (Stoyan Atanassov)

6. Find the largest possible number A having the following property: if the numbers $1, 2, \ldots, 1000$ are ordered in arbitrary way then there exist 50 consecutive numbers with sum not less than A. (Ivan Landjev)

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Grade 11

1. Find all values of the real parameter a such that the equation

$$a(\sin 2x+1)+1=(a3)(\sin x+\cos x)$$

has a solution. (Emil Kolev)

2. On the sides of an acute $\triangle ABC$ of area 1 points $A_1 \in BC$, $B_1 \in CA$ and $C_1 \in AB$ are chosen so that

$$\angle CC_1B=\angle AA_1C=\angle BB_1A=\phi,$$

where the angle ϕ is acute. The segments AA_1, BB_1 and CC_1 meet at points M, N and P.

   (a) Prove that the circumcenter of $\triangle MNP$ coincides with the orthocenter of $\triangle ABC$.

   (b) Find ϕ, if $[MNP]=2-\sqrt{3}$.

(Emil Kolev)

3. Let n be a fixed positive integer. The positive integers a, b, c and d are less than or equal to n, d is the largest one and they satisfy the equality

$$(ab+cd)(bc+ad)(ac+bd)=(da)^2(db)^2(dc)^2.$$

   (a) Prove that $d=a+b+c$.

   (b) Find the number of the quadruples (a,b,c,d) which have the required properties.

(Alexander Ivanov)

4. Find all values of the real parameter a such that the equation

$$\log_{ax}(3^x+4^x)=\log_{(ax)^2}(7^2(4^x-3^x))+\log_{(ax)^3}8^{x-1}$$

has a solution. (Emil Kolev)

5. The bisectors of $\angle BAC$, $\angle ABC$ and $\angle ACB$ of $\triangle ABC$ meet its circumcircle at points A_1, B_1 and C_1, respectively. The side AB meets the lines C_1B_1 and C_1A_1 at points M and N, respectively, the side BC meets the lines A_1C_1 and A_1B_1 at points P and Q, respectively, and the side AC meets the lines B_1A_1 and B_1C_1 at points R and S, respectively. Prove that:

   (a) the altitude of $\triangle CRQ$ through R is equal to the inradius of $\triangle ABC$;

   (b) the lines MQ, NR and SP are concurrent.

(Alexander Ivanov)

6. Prove that amongst any 9 vertices of a regular 26-gon there are three which are vertices of an isosceles triangle. Do there exist 8 vertices such that no three of them are vertices of an isosceles triangle?

(Alexander Ivanov)

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Grade 12

1. Prove that if a, b and c are integers such that the number
$$\frac{a(ab)+b(bc)+c(ca)}{2}$$
is a perfect square, then $a=b=c$. (Oleg Mushkarov)

2. Find all values of the real parameters a and b such that the graph of the function $y=x^3+ax+b$ has exactly three common points with he coordinate axes and they are vertices of a right triangle.
(Nikolai Nikolov)

3. Let $ABCD$ be a convex quadrilateral. The orthogonal projections of D on the lines BC and BA are denoted by A_1 and C_1, respectively. The segment A_1C_1 meets the diagonal AC at an interior point B_1 such that $DB_1 \geq DA_1$. Prove that the quadrilateral $ABCD$ is cyclic if and only if
$$\frac{BC}{DA_1}+\frac{BA}{DC_1}=\frac{AC}{DB_1}$$
(Nikolai Nikolov)

4. The point K on the edge AB of the cube $ABCDA_1B_1C_1D_1$ is such that the angle between the line A_1B and the plane (B_1CK) is equal to 60°. Find $\tan\alpha$, where α is the angle between the planes (B_1CK) and (ABC). (Oleg Mushkarov)

5. Prove that any triangle of area $\sqrt{3}$ can be placed into an infinite band of width $\sqrt{3}$. (Oleg Mushkarov)

6. Let m be a positive integer, $A=\{-m,-m+1,\dots,m-1,m\}$ and $f:A\to A$ be a function such that $f(f(n))=-n$ for every $n\in A$.
   (a) Prove that the number m is even.
   (b) Find the number of all functions $f:A\to A$ with the required property.

(Nikolai Nikolov)

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