Bulgarian Mathematical Olympiad 1993, IV Round

First Day

1. Find all functions f, defined and having values in the set of integer numbers, for which the following conditions are satisfied:

   (a) $f(1) = 1$;

   (b) for every two whole (integer) numbers m and n, the following equality is satisfied:

   $$f(m+n)\cdot(f(m)-f(n)) = f(m-n)\cdot(f(m)+f(n))$$

2. The point M is internal point for the triangle ABC such that: $\angle AMC = 90^\circ$, $\angle AMB = 150^\circ$ and $\angle BMC = 120^\circ$. Points P, Q and R are centers of circumscribed circles around triangles AMC, AMB and BMC. Prove that the area of triangle PQR is bigger than the area of the triangle ABC.

3. It is given a polyhedral constructed from two regular pyramids with bases heptagons (a polygon with 7 vertices) with common base $A_1A_2A_3A_4A_5A_6A_7$ and vertices respectively the points B and C. The edges BA_i, CA_i $(i = 1,\ldots,7)$, diagonals of the common base are painted in blue or red. Prove that there exists three vertices of the polyhedral given which forms a triangle with all sizes in the same color.

Second day

4. Find all natural numbers $n > 1$ for which there exists such natural numbers $a_1, a_2, \ldots, a_n$ for which the numbers $\left\{a_i + a_j \mid 1 \le i \le j \le n\right\}$ forms a full system modulo $\frac{n(n+1)}{2}$.

5. Let Oxy is a fixed rectangular coordinate system in the plane. Each ordered pair of points A_1, A_2 from the same plane which are different from O and have coordinates x_1, y_1 and x_2, y_2 respectively is associated with real number $f(A_1,A_2)$ in such a way that the following conditions are satisfied:

   (a) If $OA_1 = OB_1$, $OA_2 = OB_2$ and $A_1A_2 = B_1B_2$ then $f(A_1,A_2) = f(B_1,B_2)$.

   (b) There exists a polynomial of second degree $F(u,v,w,z)$ such that $f(A_1,A_2) = F(x_1,y_1,x_2,y_2)$.

   (c) There exists such a number $\phi \in (0,\pi)$ that for every two points A_1, A_2 for which $\angle A_1OA_2 = \phi$ is satisfied $f(A_1,A_2) = 0$.

   (d) If the points A_1, A_2 are such that the triangle OA_1A_2 is equilateral with side 1 then $f(A_1,A_2) = \frac{1}{2}$.

   Prove that $f(A_1,A_2) = \overrightarrow{OA_1}\cdot\overrightarrow{OA_2}$ for each ordered pair of points A_1, A_2.

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6. Find all natural numbers n for which there exists set S consisting of n points in the plane, satisfying the condition:

   For each point $A \in S$ there exist at least three points say X, Y, Z from S such that the segments AX, AY and AZ have length 1 (it means that $AX = AY = AZ = 1$).

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