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Problems

1.1 The Forty-Nineth IMO
Madrid, Spain, July 10–22, 2008

1.1.1 Contest Problems

First Day (July 16)

1. An acute-angled triangle $ABC$ has orthocenter $H$. The circle passing through $H$ with center the midpoint of $BC$ intersects the line $BC$ at $A_1$ and $A_2$. Similarly, the circle passing through $H$ with center the midpoint of $CA$ intersects the line $CA$ at $B_1$ and $B_2$, and the circle passing through $H$ with center the midpoint of $AB$ intersects the line $AB$ at $C_1$ and $C_2$. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

2. (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers $x, y, z$, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers $x, y, z$, each different from 1, satisfying $xyz = 1$.

3. Prove that there exist infinitely many positive integers $n$ such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

Second Day (July 17)

4. Find all functions $f : (0, +\infty) \rightarrow (0, +\infty)$ (so, $f$ is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers $w, x, y, z$, satisfying $wx = yz$. 
5. Let \( n \) and \( k \) be positive integers with \( k \geq n \) and \( k - n \) an even number. Let \( 2n \) lamps labelled 1, 2, \ldots, \( 2n \) be given, each of which can be either on or off. Initially all the lamps are off. We consider sequence of steps: at each step one of the lamps is switched (from on to off or from off to on).
Let \( N \) be the number of such sequences consisting of \( k \) steps and resulting in the state where lamps 1 through \( n \) are all on, and lamps \( n + 1 \) through \( 2n \) are all off.
Let \( M \) be the number of such sequences consisting of \( k \) steps and resulting in the state where lamps 1 through \( n \) are all on, and lamps \( n + 1 \) through \( 2n \) are all off, but where none of the lamps \( n + 1 \) through \( 2n \) is ever switched on.
Determine the ratio \( N/M \).

6. Let \( ABCD \) be a convex quadrilateral with \(|BA| \neq |BC|\). Denote the incircles of triangles \( ABC \) and \( ADC \) by \( \omega_1 \) and \( \omega_2 \) respectively. Suppose that there exists a circle \( \omega \) tangent to the ray \( BA \) beyond \( A \) and to the ray \( BC \) beyond \( C \), which is also tangent to the lines \( AD \) and \( CD \). Prove that the common external tangents of \( \omega_1 \) and \( \omega_2 \) intersect on \( \omega \).

1.1.2 Shortlisted Problems

1. **A1 (KOR) IMO** Find all functions \( f : (0, +\infty) \to (0, +\infty) \) (so, \( f \) is a function from the positive real numbers to the positive real numbers) such that
\[
\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}
\]
for all positive real numbers \( w, x, y, z \), satisfying \( wx = yz \).

2. **A2 (AUT) IMO**
   (a) Prove that
   \[
   \frac{x^2}{(x - 1)^2} + \frac{y^2}{(y - 1)^2} + \frac{z^2}{(z - 1)^2} \geq 1
   \]
   for all real numbers \( x, y, z \), each different from 1, and satisfying \( xyz = 1 \).
   (b) Prove that equality holds above for infinitely many triples of rational numbers \( x, y, z \), each different from 1, satisfying \( xyz = 1 \).

3. **A3 (NET)** Let \( S \subseteq \mathbb{R} \) be a set of real number. We say that a pair \( (f, g) \) of functions from \( S \) to \( S \) is a Spanish Couple on \( S \), if they satisfy the following conditions:
   (i) Both functions are strictly increasing, i.e. \( f(x) < f(y) \) and \( g(x) < g(y) \) for all \( x, y \in S \) with \( x < y \);
   (ii) The inequality \( f(g(g(x))) < g(f(x)) \) holds for all \( x \in S \).
Decide whether there exists a Spanish Couple
   (a) on the set \( S = \mathbb{N} \) of positive integers;
   (b) on the set \( S = \{a - 1/b : a, b \in \mathbb{N}\} \).
4. **A4 (AUT)** For an integer \( m \), denote by \( t(m) \) the unique number in \{1, 2, 3\} such that \( m + t(m) \) is a multiple of 3. A function \( f : \mathbb{Z} \to \mathbb{Z} \) satisfies \( f(-1) = 0 \), \( f(0) = 1 \), \( f(1) = -1 \) and

\[
f(2^n + m) = f(2^n - t(m)) - f(m)
\]
for all integers \( m, n \geq 0 \) with \( 2^n > m \).

Prove that \( f(3p) \geq 0 \) holds for all integers \( p \geq 0 \).

5. **A5 (SVK)** Let \( a, b, c, d \) be positive real numbers such that \( abcd = 1 \) and \( a + b + c + d > \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \).

Prove that

\[
a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}.
\]

6. **A6 (LIT)** Let \( f : \mathbb{R} \to \mathbb{N} \) be a functions which satisfies

\[
f(x + \frac{1}{f(y)}) = f(y + \frac{1}{f(x)}), \quad \text{for all } x, y \in \mathbb{R}.
\]

Prove that there is a positive integer which is not a value of \( f \).

7. **A7 (GER)** Prove that for any four positive real numbers \( a, b, c, d \) the inequality

\[
\frac{(a-b)(a-c)}{a+b+c} + \frac{(b-c)(b-d)}{b+c+d} + \frac{(c-d)(c-a)}{c+d+a} + \frac{(d-a)(d-b)}{d+a+b} \geq 0
\]
holds. Determine all cases of equality.

8. **C1 (NET)** A box is a rectangle in the plane whose sides are parallel to the coordinate axes and have positive lengths. Two boxes intersect if they have a common point in their interior or on their boundary. Find the largest \( n \) for which there exist \( n \) boxes \( B_1, \ldots, B_n \) such that \( B_i \) and \( B_j \) intersect if and only if \( i \equiv j \pm 1 \pmod{n} \).

9. **C2 (SER)** For every positive integer \( n \) determine the number of permutations \( (a_1, \ldots, a_n) \) of the set \{1, 2, \ldots, n\} with the following property:

\[
2(a_1 + a_2 + \cdots + a_k) \text{ is divisible by } k \text{ for } k = 1, 2, \ldots, n.
\]

10. **C3 (PER)** Consider the set \( S \) of all points with integer coordinates in the coordinate plane. For a positive integer \( k \), two distinct points \( A, B \in S \) will be called \( k \)-friends if there is a point \( C \in S \) such that the area of the triangle \( ABC \) is equal to \( k \). A set \( T \subseteq S \) will be called a \( k \)-clique if every two points in \( T \) are \( k \)-friends. Find the least positive integer \( k \) for which there exists a \( k \)-clique with more than 200 elements.

11. **C4 (FRA) IMO** Let \( n \) and \( k \) be positive integers with \( k \geq n \) and \( k - n \) an even number. Let \( 2n \) lamps labelled 1, 2, \ldots, 2n be given, each of which can be either
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on or off. Initially all the lamps are off. We consider sequence of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let $N$ be the number of such sequences consisting of $k$ steps and resulting in the state where lamps 1 through $n$ are all on, and lamps $n+1$ through $2n$ are all off.

Let $M$ be the number of such sequences consisting of $k$ steps and resulting in the state where lamps 1 through $n$ are all on, and lamps $n+1$ through $2n$ are all on, but where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine the ratio $N/M$.

12. **C5 (RUS)** Let $S = \{x_1, x_2, \ldots, x_{k+l}\}$ be a $k + l$-element set of real numbers contained in the interval $[0, 1]$; $k$ and $l$ are positive integers. A $k$-element subset $A \subseteq S$ is called nice if

$$\frac{1}{k} \sum_{x_j \in A} x_j - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \leq \frac{k + l}{2kl}.$$

Prove that the number of nice subsets is at least $\frac{2}{k+l} \binom{k+l}{k}$.

13. **C6 (NET)** For $n \geq 2$, let $S_1, S_2, \ldots, S_{2^n}$ be $2^n$ subsets of $A = \{1, 2, 3, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indeces $a$ and $b$ with $a < b$ and elements $x, y, z \in A$ with $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Prove that at least one of the sets $S_1, S_2, \ldots, S_{2^n}$ contains no more than $4n$ elements.

14. **G1 (RUS)** IMO1 An acute-angled triangle $ABC$ has orthocenter $H$. The circle passing through $H$ with center the midpoint of $BC$ intersects the line $BC$ at $A_1$ and $A_2$. Similarly, the circle passing through $H$ with center the midpoint of $CA$ intersects the line $CA$ at $B_1$ and $B_2$, and the circle passing through $H$ with center the midpoint of $AB$ intersects the line $AB$ at $C_1$ and $C_2$. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

15. **G2 (LUX)** Given a trapezoid $ABCD$ with parallel sides $AB$ and $CD$, assume that there exist points $E$ on line $BC$ outside the segment $BC$, and $F$ inside the segment $AD$, such that $\angle DAE = \angle CBF$. Denote by $I$ the intersection point of $CD$ and $EF$, and by $J$ the intersection point of $AB$ and $EF$. Let $K$ be the midpoint of the segment $EF$. Assume $K$ does not line on the lines $AB$ and $CD$.

Prove that $I$ belongs to the circumcircle of $\triangle ABK$ if and only if $K$ belongs to the circumcircle of $\triangle CDJ$.

16. **G3 (PER)** Let $ABCD$ be a convex quadrilateral and let $P$ and $Q$ be the points such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point $E$ on the line segment $PQ$ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QBE$. Show that the quadrilateral $ABCD$ is cyclic.

17. **G4 (IRN)** Let $BE$ and $CF$ be the altitudes in an acute triangle $ABC$. Two circles passing through the points $A$ and $F$ are tangent to the line $BC$ at the points $P$ and $Q$ so that $B$ lies between $C$ and $Q$. Prove that the lines $PE$ and $QF$ intersect on the circumcircle of $\triangle AEF$. 

18. **G5 (NET)** Let \( k \) and \( n \) be integers with \( 0 \leq k \leq n - 2 \). Consider a set \( L \) of \( n \) lines in the plane such that no two of them are parallel and no three have a common point. Denote by \( I \) the set of intersection points of lines in \( L \). Let \( O \) be a point in the plane not lying on any line of \( L \).

A point \( X \in I \) is colored in red if the open line segment \((OX)\) intersects at most \( k \) lines from \( L \). Prove that \( I \) contains at least \( \frac{1}{2}(k+1)(k+2) \) red points.

19. **G6 (SER)** Let \( ABCD \) be a convex quadrilateral. Prove that there exists a point \( P \) inside the quadrilateral such that

\[ \angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ \]

if and only if the diagonals \( AC \) and \( BD \) are perpendicular.

20. **G7 (RUS)** Let \( AB \) be a convex quadrilateral with \( |BA| \neq |BC| \). Denote the incircles of triangles \( ABC \) and \( ADC \) by \( \omega_1 \) and \( \omega_2 \) respectively. Suppose that there exists a circle \( \omega \) tangent to the ray \( BA \) beyond \( A \) and to the ray \( BC \) beyond \( C \), which is also tangent to the lines \( AD \) and \( CD \). Prove that the common external tangents of \( \omega_1 \) and \( \omega_2 \) intersect on \( \omega \).

21. **N1 (AUS)** Let \( n \) be a positive integer and let \( p \) be a prime number. Prove that if \( a, b, c \) are integers (not necessarily positive) satisfying the equations

\[ a^n + pb = b^n + pc = c^n + pa, \]

then \( a = b = c \).

22. **N2 (IRN)** Let \( a_1, a_2, \ldots, a_n \) be distinct positive integers, \( n \geq 3 \). Prove that there exist distinct indeces \( i \) and \( j \) such that \( a_i + a_j \) does not divide any of the numbers \( 3a_1, 3a_2, \ldots, 3a_n \).

23. **N3 (IRN)** Let \( a_0, a_1, a_2 \) be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term, i.e. \( (a_i, a_{i+1}) > a_{i-1} \) for all \( i \geq 1 \). Prove that \( a_n \geq 2^n \) for all \( n \geq 0 \).

24. **N4 (SER)** Let \( n \) be a positive integer. Show that the numbers

\[ \begin{pmatrix} 2^n - 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2^n - 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2^n - 1 \\ 2 \end{pmatrix}, \ldots, \begin{pmatrix} 2^n - 1 \\ 2^{n-1} - 1 \end{pmatrix} \]

are congruent modulo \( 2^n \) to \( 1, 3, 5, \ldots, 2^n - 1 \) in some order.

25. **N5 (FRA)** For every \( n \in \mathbb{N} \) let \( d(n) \) denote the number of (positive) divisors of \( n \). Find all functions \( f : \mathbb{N} \to \mathbb{N} \) with the following properties:

(i) \( d(f(x)) = x \) for all \( x \in \mathbb{N} \);

(ii) \( f(xy) \) divides \( (x-1)y^{y-1}f(x) \) for all \( x, y \in \mathbb{N} \).

26. **N6 (LIT)** Prove that there exist infinitely many positive integers \( n \) such that \( n^2 + 1 \) has a prime divisor which is greater than \( 2n + \sqrt{2n} \).
Solutions
2.1 Solutions to the Shortlisted Problems of IMO 2008

1. For \( x = y = z = w \) the functional equation gives \( f(x)^2 = f(x^2) \) for all \( x \in \mathbb{R}_+ \). In particular, \( f(1) = 1 \). Setting \( \sqrt{w}, \sqrt{x}, \sqrt{y}, \sqrt{z} \) in the equation yields

\[
\frac{f(w) + f(x)}{f(y) + f(z)} = \frac{w + x}{y + z}, \quad \text{whenever } wx = yz.
\]

Choosing \( z = 1 \) we get \( w = y/x \) and \( f(y/x) + f(x) = (\frac{x}{w} + x) \cdot \frac{f(x)}{x^2} \). Now if we place \( y = x^2 \) we get \( f(x) = x \cdot \frac{f(x)^2 + 1}{x^2 + 1} \) which is equivalent to \( (f(x) - x)(f(x) - \frac{1}{x}) = 0 \). Assume that there are \( x, y \in \mathbb{R}_+ \setminus \{1\} \) such that \( f(x) = x \) and \( f(w) = \frac{1}{w} \).

Choosing \( y = z = \sqrt{wx} \) and placing in the equation implies \( \frac{1}{w} + x = (w + x) \cdot \frac{f(\sqrt{wx})}{\sqrt{wx}} \). If \( f(\sqrt{wx}) = \sqrt{wx} \) then we have \( w = 1 \). Otherwise, if \( f(\sqrt{wx}) = 1/\sqrt{wx} \) we get \( x = 1 \), contrary to our assumption. Therefore we either have \( f(x) = x \) for all \( x \in \mathbb{R}_+ \) or \( f(x) = \frac{1}{x} \) for all \( x \in \mathbb{R}_+ \). It is easy to verify that both functions satisfy the original equation.

2. (a) Substituting \( a = \frac{x}{x - 1} \), \( b = \frac{x}{x - 1} \), \( c = \frac{x}{x - 1} \) the inequality becomes equivalent to \( a^2 + b^2 + c^2 \geq 1 \) while the constraint becomes \( a + b + c = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1 \).

The last equation is equivalent to \( 2(a + b + c) = (a + b + c)^2 - (a^2 + b^2 + c^2) + 2 = (a + b + c)^2 + 1 - [(a^2 + b^2 + c^2) - 1] \), or \( [(a + b + c)^2 - 1] = (a^2 + b^2 + c^2) - 1 \) which immediately implies that \( a^2 + b^2 + c^2 \geq 1 \).

(b) The equality holds if and only if \( a + b + c = 1 \) and \( a + b + c = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1 \). Expressing \( c = 1 - a - b \) yields \( -ab + a + b - a^2 - b^2 = 0 \). It suffices to prove that there are infinitely many rational numbers \( a \) for which the quadratic equation \( b^2 - (a - 1)b - a(a - 1) = 0 \) has a rational solution \( b \). The last quadratic equation has a rational solution if and only if its discriminant \( (a - 1)^2 - 4a(a - 1) = (1 - a)(1 + 3a) \) is a square of a rational number. We want to find infinitely many rational numbers \( \frac{a}{b} \) such that \( (1 - \frac{a}{b}) \cdot \left(1 + 3 \frac{a}{b}\right) \) is a square of a rational number, which is equivalent to \( (q - p)(q + 3p) \) being a square of an integer. However, for each \( m, n \in \mathbb{N} \) the system \( q - p = (2m + 1)^2 \) and \( q + 3p = 2n + 1 \) has a solution \( p = n^2 + n + m^2 - m, q = n^2 + 3m^2 + 3m + 1 \). Keeping \( m \) fixed, and increasing \( n \) would guarantee that we are getting infinitely many different fractions \( a = \frac{p}{q} \).

3. (a) Assume that \((f, g)\) is a Spanish Couple on \( \mathbb{N} \). If \( g(a) \geq g(b) \) then \( a > b \) \((a \leq b \) would yield \( g(a) \leq g(b))\). Let us introduce the notation

\[
g^k(x) = g(g(\cdots g(x) \cdots)),
\]

If we assume that \( g(x) < x \) for some \( x \in \mathbb{N} \) we get \( g(g(x)) < g(x) < x \), and by induction \( (g^k(x))^\mathbb{N}_{k=1} \) is infinite decreasing sequence from \( \mathbb{N} \) which is impossible. Hence \( g(x) \geq x \) for all \( x \in \mathbb{N} \). The same holds for \( f \). If for some
5. Using the inequality between arithmetic and geometric mean we get:

\[
\frac{3}{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{d}{a}\right)} + \left(\frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}\right) = \left(\frac{2a}{b} + \frac{b}{c} + \frac{c}{a}\right) + \left(\frac{2b}{c} + \frac{c}{d} + \frac{d}{a}\right) + \left(\frac{2c}{d} + \frac{d}{a} + \frac{a}{b}\right) + \left(\frac{2d}{a} + \frac{a}{b} + \frac{b}{c}\right)
\]

\[
\geq 4\sqrt[3]{\frac{a^3}{bcd}} + 4\sqrt[3]{\frac{b^3}{cda}} + 4\sqrt[3]{\frac{c^3}{dab}} + 4\sqrt[3]{\frac{d^3}{abc}} = 4(a + b + c + d)
\]

\[
> 3\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) + (a + b + c + d).
\]

The required inequality now follows immediately.
6. Assume the contrary, that \( f \) is onto. There exists a sequence of numbers \( a_n \) such that \( f(a_n) = n \).

If \( f(u) = f(v) \) then \( f(u + 1/n) = f(a_n + 1/f(u)) = f(a_n + 1/f(v)) = f(v + 1/n) \)
for all \( n \in \mathbb{N} \) and by induction \( f(u + m/n) = f(v + m/n) \) for all \( m,n \in \mathbb{N} \).

Applying this to \( u = a_1, v = a_1 + 1/f(a_1 - 1) \), and \( n = f(a_1 - 1) \) we get \( 1 = f(a_1) = f(a_1 + 1/n) = f(a_1 + 2/n) = \cdots = f(a_1 + 1) \). This further implies that \( f(a_1 + q + 1) = f(a_1 + q) \) for all rational numbers \( q \).

For fixed \( y \) we have

\[
\Gamma(y) = \left\{ f \left(y + \frac{1}{n}\right) : n \in \mathbb{N} \right\} = \left\{ f \left(\frac{1}{f(x)}\right) : x \in \mathbb{R} \right\}
= \left\{ f \left(x + \frac{1}{f(y)}\right) : x \in \mathbb{R} \right\} = \mathbb{N}.
\]

Particularly, \( \Gamma(a_1) = \mathbb{N} \) so we could assume that \( a_2, a_3, \ldots \) are chosen from \( \Gamma(a_1) \). Hence for each \( n \geq 2 \) there exists \( k_n \in \mathbb{N} \) such that \( a_n = a_1 + 1/k_n \).

Now we have \( f(a_1 + 1/n) = f(a_1 + 1/f(a_1)) = f(a_n + 1) = f(a_1 + \frac{1}{k_1} + 1) = f(a_1 + 1/k_1) = f(a_n) = n \).

Since \( \Gamma(a_1 + \frac{1}{k}) = \mathbb{N} \) there exists \( d \) such that \( f(a_1 + \frac{1}{k}) + \frac{1}{k} = 1 \). Assume that \( \frac{1}{k} + \frac{1}{d} = \frac{p}{q} \) for relatively prime numbers \( p \) and \( q \). Since \( \frac{1}{k} + \frac{1}{d} \neq 1 \) we have \( q > 1 \).

Let \( k \) be an integer such that \( kp \equiv 1 \mod{q} \). Then \( 1 = f(a_1 + p/q) = f(a_1 + kp/q) = f(a_1 + 1/q) = q \) which is a contradiction.

7. The left-hand side can be rewritten as

\[
L = \frac{(a - c)^2}{a + b + c} + \frac{(b - d)^2}{b + c + d} + (a - c)(b - d) \cdot \left( \frac{2d + b}{(b + c + d)(d + a + b)} - \frac{2c + a}{(a + b + c)(c + d + a)} \right).
\]

In order to prove that \( L \geq 0 \) it suffices to establish the following inequality:

\[
(a - c)(b - d) \left( \frac{2d + b}{(b + c + d)(d + a + b)} - \frac{2c + a}{(a + b + c)(c + d + a)} \right) \geq \frac{-2 |(a - c)(b - d)|}{\sqrt{a + b + c} \cdot \sqrt{b + c + d}}
\]

If \( a = c \) or \( b = d \) the inequality is obvious. Assume that \( a > c \) and \( b > d \). Our goal is to prove that

\[
\left| \frac{2d + b}{(b + c + d)(d + a + b)} - \frac{2c + a}{(a + b + c)(c + d + a)} \right| \leq \frac{2}{\sqrt{a + b + c} \cdot \sqrt{b + c + d}}.
\]

Both fractions on the left-hand side are positive, hence it is enough to prove that each of them is smaller than the right-hand side. These two inequalities are analogous, so let us prove the first one. After squaring both sides, cross-multiplying, and subtracting we get:
Let us call a permutation nice hence $x_n$ if it satisfies the stated property. We want to calculate the number $x_n$ of nice permutations. For $n = 1, 2, 3$ every permutation is nice hence $x_n = n!$ for $n \leq 3$.

Assume now that $n \geq 4$. From $(n-1) \mid 2(a_1 + \cdots + a_{n-1}) = 2[(1+\cdots+n) - a_n] = n(n+1) - 2a_n = (n+2)(n-1) - 2(a_n - 1)$ we conclude that $(n-1) \mid 2(a_n - 1)$.

Let us prove that $a_n \in \{1, n\}$ for odd $n$. Assume the contrary. Then $n - 1 = 2(a_n - 1)$, i.e. $a_n = \frac{n+1}{2}$. Then $n - 2 \mid 2(a_1 + \cdots + a_{n-2}) = n(n-1) - 2a_n - 2a_{n-1} = n(n+1) - (n+1) - 2a_{n-1} = (n-2)(n+2) + 3 - 2a_{n-1}$ which gives $n - 2 \mid 2a_{n-1} - 3$. Since $\frac{2a_{n-1} - 3}{n-2} \leq \frac{2(n-3)}{n-2} = 1 + \frac{1}{n-2} < 2$ we get $n - 2 = 2a_{n-1} - 3.$

This implies that $a_{n-1} = \frac{n-1}{2} = a_n$ which is a contradiction.

Therefore, for $n \geq 4$ we must have $a_n \in \{1, n\}$. There are $x_{n-1}$ nice permutations for $a_n = n$, and for $a_n = 1$, the problem reduces to counting the nice permutations of the set $\{2, 3, \ldots, n\}$ satisfying the given property. However, since $2(a_1 + \cdots + a_k) = 2k + 2((a_1 - 1) + \cdots (a_k - 1))$ we get $k \mid 2(a_1 + \cdots + a_k)$ if and
10. Since the area of a triangle $ABC$ is equal to $\frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$ we have that $(0,0)$ and $(a,b)$ are $k$-friends if and only if there exists a point $(x,y)$ such that $ay - bx = \pm 2k$. According to the Euclid’s algorithm such integers $x$ and $y$ will exist if and only if $\gcd (a,b) | 2k$. Similarly $(a,b)$ and $(c,d)$ are $k$-friends if and only if $\gcd (c-a,d-b) | 2k$.

Assume that there exists a $k$-clique $S$ of size $n^2 + 1$ for some $n \geq 1$. Then there are two elements $(a,b), (c,d) \in S$ such that $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$. This implies $n | \gcd (a-c,b-d) | 2k$, or equivalently, $n | 2k$.

Therefore, for a $k$-clique of size 200 to exist, we must have $n | 2k$ for all $n \in \{1,2,\ldots,14\}$. Therefore $k \geq 4 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 180180$.

It is easy to see that all lattice points from the square $[0,14]^2$ are 180180-friends.

11. The number of sequences in which the lamp $i$ is switched (on or off) exactly $\alpha_i$ times ($i = 1,2,\ldots,2n$) is equal to $\frac{k!}{\alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_{2n}!}$. Therefore

$$M = k! \sum_{\alpha_1 + \cdots + \alpha_n = k, 2 \nmid \alpha_1, 2 \nmid \alpha_2, \ldots, 2 \nmid \alpha_n} \frac{1}{\alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_{2n}!},$$

Similarly we get

$$N = k! \sum_{\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n = k, 2 \nmid \alpha_1, 2 \nmid \alpha_2, \ldots, 2 \nmid \alpha_n, 2 \nmid \beta_1, \ldots, 2 \nmid \beta_n} \frac{1}{\alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_{2n}! \cdot \beta_1! \cdot \ldots \cdot \beta_{2n}!},$$

where the summation is over all possible $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ that satisfy $\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n = k, 2 | \alpha_1, \ldots, 2 | \alpha_n, 2 | \beta_1, \ldots, 2 | \beta_n$. We see that the sum in (1) is equal to the coefficient of $X^k$ in the expansion

$$f(X) = \left( X + \frac{X^3}{3!} + \frac{X^5}{5!} + \cdots \right)^n = \sinh^n(X)$$

while the sum in (2) is equal to the coefficient of $X^k$ in the expansion

$$g(X) = \left( X + \frac{X^3}{3!} + \frac{X^5}{5!} + \cdots \right)^n \cdot \left( 1 + \frac{X^2}{2!} + \frac{X^4}{4!} + \cdots \right)^n$$

$$= \sinh^n(X) \cdot \cosh^n(X) = \frac{1}{2^n} \sinh^n(2X).$$

Assume that $\sinh^n(X) = \sum_{i=0}^{\infty} a_i X^i$ for some real numbers $a_1, a_2, \ldots$. Then we have $\sinh^n(2X) = \sum_{i=0}^{\infty} a_i (2X)^i = \sum_{i=0}^{\infty} a_i \cdot 2^i \cdot X^i$. Therefore

$$N \quad M = \frac{k! \cdot a_k}{2^k \cdot k! \cdot a_k \cdot 2^k} = 2^{n-k}.$$
14. Let \( \pi = (\pi_1, \ldots, \pi_{k+l}) \) of \( S \) consider the sets \( A_i^k, A_i^{k+1}, \ldots, A_i^{k+l} \) defined as \( A_i^k = \{ \pi_i, \pi_{i+1}, \ldots, \pi_{i+k-1} \} \) (index is modulo \( k+l \)). We will prove that at least two of the sets \( A_i^k \) are nice. Let us paint the sets \( A_i^k \) red and green in the following way: \( A_i^k \) is green if and only if \( \Gamma(A_i^k) \geq m \). Notice that \( \Gamma(A_i^k) - \Gamma(A_{i+1}^k) = \frac{1}{k} (\pi_i - \pi_{i+k}) \) is of absolute value \( \leq \frac{1}{k} \). Therefore \( \Gamma(A_i^k) - \pi \leq \frac{1}{k} \Gamma \). Hence whenever two consecutive sets in the sequence \( A_i^k, \ldots, A_i^{k+l} \) are of different colors one of them must be nice. If there are \( \geq 2 \) sets of each of the colors, it is obvious that at least two of the sets will be nice. Assume that there is only one red set and that it is the only nice set. Without loss of generality assume that \( A_i^k \) is red. Then \( m(k+1) = \Gamma(A_i^k) + \Gamma(A_i^{k+1}) + \cdots + \Gamma(A_i^{k+l}) \geq m + \frac{1}{k^2} \cdots + m + \frac{1}{k} = (k+l)m \) which is a contradiction. Now we can prove the required statement. To each of \((k+l)!烟囱\) permutations of \( S \) we assign at least two nice sets. Each set is counted \((k+l)!烟囱\) times so there are at least \( \frac{2(k+l)!}{k+1} \cdot \frac{1}{k+1} \) nice sets.

13. We will prove a stronger result, that for each \( k \) we have

\[
\sum_{i=1}^{k} \left( |S_i| - (n+1) \right) \leq (2n+1) \cdot 2^{n-1}. \tag{1}
\]

The desired statement follows from (1) because if \( |S_i| \geq 2n + 2 \) for each \( i \) then \((2n+1) \cdot 2^{n-1} \geq 2^n \cdot (n+1) \) which is impossible.

We will use induction on \( n \) to prove (1). First, for \( n = 1 \) and any subsets \( S_1, S_2, \ldots, S_k \) of \( \{1, 2, 3, 4\} \) we want to prove \(|S_1| + \cdots + |S_k| \leq 2^0 \cdot 3 + k(1+1) = 3 + 2k \). It suffices to verify this only when \(|S_i| \geq 3 \) for each \( i \). If there is one set with four elements, then \( k = 1 \) and the inequality is satisfied. If all sets have cardinality 3, then \( k \leq 3 \) (\( \{1, 3, 4\} \) and \( \{2, 3, 4\} \) can’t be both among the chosen sets), hence \( 3k \leq 6 + 2k \).

Assume now that the statement is true for \( n - 1 \). Let us divide the subsets \( S_1, \ldots, S_k \) of \( \{1, 2, \ldots, 2^{n+1}\} \) in two families: \( \mathcal{A} = \{ A_1, \ldots, A_l \} \) – those with all elements greater than \( 2^n \), and \( \mathcal{B} = \{ B_1, \ldots, B_{k-l} \} \) – the remaining subsets. Using the inducational hypothesis we obtain \( \sum_{i=1}^{l} (|A_i| - (n+1)) \leq (2n-1) \cdot 2^{n-2} - l \). Let us denote by \( a_k \) the smallest element of \( B_1 \cap \{ 2^n + 1, \ldots, 2^{n+1} \} \) if it exists. Let \( H_j = B_j \cap \{ 1, \ldots, 2^n \} \) and \( G_j = B_j \cap \{ 2^n + 1, \ldots, 2^{n+1} \} \). If \( i < j \) we claim that \( G_{j} \cap G_{j} = \emptyset \). If not, considering \( z \in G_{j} \cap G_{j} \) and taking \( y = a_i, x \in H_{j} \) we get a contradiction since \( x < y < z \) and \( x, z \in B_{j}, y \in B_{i} \). Sets \( G_j \) are disjoint, and the inducational hypothesis holds for sets \( H_i \), hence

\[
\sum_{i=1}^{k-l} (|B_i| - (n+1)) = \sum_{i=1}^{l} (|H_i| - n) + \sum_{i=1}^{k-l} (|G_i| \leq (2n-1)2^{n-2} + 2^n. \text{Thus} \quad \sum_{i=1}^{k} (|S_i| - (n+1)) \leq (2n-1) \cdot 2^{n-2} - 2 - l + 2^n \leq (2n+1)2^{n-1}.
\]

14. Let \( A', B', C' \) be the midpoints of \( BC, CA, AB \), respectively, and \( A'', B'', C'' \) be the midpoints of \( HA, HB, HC \) respectively. Let \( O \) be the circumcenter of \( \triangle ABC \) and \( R \) its circumradius. Pythagoras theorem implies \( OA_1^2 = OA_2^2 + A'A_1^2 = \)

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2 Solutions

$O A'^2 + A'H^2$. Since $H A' O A''$ is a parallelogram we have that $O A'^2 + A'H^2 = 4(2 + A'H^2)$. However, since $A'' H A$ is a parallelogram we have that $A'H = O A = R$. Thus $O A'^2 = \frac{1}{4}(R^2 + OH^2)$. Similar relations for $O A_2, O B_1, O B_2, O C_1, O C_2$ imply that the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle with center $O$.

15. Assume that the distribution of the points is such that $ABEF$ is a convex quadrilateral and $C$ belongs to the segment $BE$ (other cases are analogous).

Let $JF = p$, $FL = q$, $IK = r$. Then $K E = q + r$. Let us further denote $D I = s$, $I C = t$, $J A = x$, $A B = y$. Since $ABEF$ is cyclic we have $J A \cdot J B = J F \cdot J E$, i.e. $x(x + y) = p(p + 2q + 2r)$. From $C D | J B$ we have $\frac{y}{q} = \frac{x}{p}$ and $\frac{r}{y} = \frac{x}{p + 2q + 2r}$. The last three equalities imply that $s t = q(q + 2r)$.

The quadrilateral $ABKI$ is cyclic if and only if $x(x + y) = (p + q)(p + q + r)$. $J C K D$ is cyclic if and only if $(p + q)r = s t$. We want to prove that

$$x(x + y) = (p + q)(p + q + r) \iff (p + q)r = st.$$ 

Using the equalities we already have, we can eliminate $x, x + y, s$, and $t$ from the previous equivalence. Hence it suffices to prove that:

$$\begin{bmatrix}
p(p + 2q + 2r) \\
(p + q)(p + q + r)
\end{bmatrix} = 1 \iff \frac{(p + q)r}{q(q + 2r)} = 1 \iff \frac{[p(p + 2q + 2r)]}{[p(p + 2q + 2r)]} = \frac{(p + q)(p + q + r)}{(p + q)r} = 1 = \frac{q(q + 2r)}{q(q + 2r)}.$$ 

The last equivalence becomes obvious once we multiply all the terms.

16. Let us first consider the case $E Q \neq E P$. Assume that $E Q < E P$ and denote by $A'$ and $D'$ the intersections of $E A$ and $E D$ with the circumcircle of $A P Q D$. Then $\angle P A A' = \angle P D A' + \angle D' A A' = \angle P A D + \angle D' A A'$ while $\angle Q D D' = \angle Q D A' + \angle A' D D'$ hence $\angle Q D A' = \angle P A D'$. This means that $A' Q = P D'$ and $A' D' \| Q P$. Therefore $\angle D E Q = \angle D A' A' = \angle D A A'$ hence $Q E$ is a tangent to the circumcircle of $\triangle D A E$. Let $M$ be the intersection of $A D$ and $P Q$. Then $M E^2 = M D \cdot M A$. Since $A P Q D$ is cyclic we have that $M D \cdot M A = M Q \cdot M P$ hence $M E^2 = M Q \cdot M P$. Assume that $B C$ intersects $P Q$ at a point $N$. Then $N E^2 = N Q \cdot N P$, and since there is the unique point $X$ on the line $P Q$ for which $X E^2 = X Q \cdot X P$ we conclude that $M \equiv N$. Now from $M D \cdot M A = M E^2 = M C \cdot M B$ we get that $A B C D$ is cyclic.

If $E Q = E P$ then it is easy to prove that the perpendicular bisectors of $A D, B C, P Q$ coincide hence $A B C D$ is an isosceles trapezoid hence it is cyclic.

17. Let $M$ be the intersection point of $Q F$ and $P E$. We need to prove that $\angle Q M P = \angle B A C$. Since $\angle M Q P = \angle Q A B$ ($Q B$ is a tangent to the circle around $\triangle Q F A$) it is enough to prove that $\angle Q A B + \angle B A C = \angle Q M P + \angle M Q P$, or, equivalently $\angle Q A E = \angle E P C$. Therefore we need to prove that $A Q P E$ is a cyclic quadrilateral. From $B Q^2 = B F \cdot B A = B P^2$ we get $B P = B Q$. Adding $B F \cdot B A = B P^2$ to $A F \cdot A E$.
18. We will use the induction on $P$. Assume first that there exists a point $L$ through $(i)$ that contains the points $P_1, P_2, \ldots, P_n \in I$, while the other ray contains the points $Q_1, \ldots, Q_{n-1} \in I$. Assume that $P_i$ are sorted according to their distance from $P$, and the same holds for $Q_i$. Consider the open segments $(OP_i)$ and $(OQ_i)$. Each line not containing any of $P_i$ and $P_i+1$ must intersect either both or none of these segments. The line passing through $P_i$ (other than $l$) could intersect $(OP_i)$ and similar fact holds for the line passing through $P_i+1$. Hence the number of intersections of $(OP_i)$ and $(OQ_i)$ with lines from $L$ differ by at most 1. Therefore $P_1, P_2, \ldots, P_{\min(k,n)}$ are all red. Similar holds for $OQ_i$ hence there are at least $k+1$ red points on $l$.

If we remove $l$ together with $n-1$ points on it, the remaining configuration allows us to apply the inductive hypothesis. There are at least $\frac{1}{2}k\cdot(k+1)$ points $G$ from $I\setminus\{l\}$ for which $(OG)$ intersects at most $k-1$ lines from $L\setminus\{l\}$. Therefore there are at least $\frac{1}{2}k\cdot(k+1)+k+1 = \frac{1}{2}(k+1)(k+2)$ red points.

19. Assume first that there exists a point $P$ inside $ABCD$ with the described property.

Let $K, L, M, N$ be the feet of perpendiculars from $P$ to $AB, BC, CD$, and $DA$ respectively. We have $\angle KNM = \angle KNP + \angle PNM = \angle KAP + \angle PDM = 9^\circ$ and similarly $\angle KLM = \angle LMN = 9^\circ$ hence $KLMN$ is a rectangle. Denote by $W, X, Y, Z$ the feet of perpendiculars from $P$ to $KL, LM, MN, NK$. From $\triangle PLX \sim \triangle PCM$ we get that $CM = \frac{XL}{PX} \cdot PM = PM \cdot \frac{PW}{PX}$. Similarly $DM = PM \cdot \frac{PW}{PY}, CL = PL \cdot \frac{PX}{PW}, BL = PL \cdot \frac{PZ}{PW}$. Notice that

$$CM : DM = \frac{PW \cdot PY}{PZ \cdot PX} = CL : BL$$

hence $BD \parallel ML$. Similarly $AC \parallel KL$ hence $AC \perp BD$.

Conversely, assume that $ABCD$ is a convex quadrilateral for which $AC \perp BD$. Let $P'$ be any point in the plane and consider a triangle $M'P'L'$ for which $M'L' \parallel BD, P'M' \perp CD$, and $P'L' \perp CB$. Let $K'$ be the point for which $P'K' \perp AB$ and $K'L' \parallel AC$. Let $N'$ be the point such that $K'L'M'N'$ is a rectangle. Consider the four lines $\alpha_k, \alpha_t, \alpha_m, \alpha_n$ through $K', L', M', N'$ perpendicular to $P'K', P'L', P'M'$, and
20. Let $M = \alpha_k \cap \alpha_i$, $B' = \alpha_k \cap \alpha_i$, $C' = \alpha_i \cap \alpha_m$, and $D' = \alpha_m \cap \alpha_i$. Using the previously established result we have: $A'C' \parallel K'L'$ and $B'D' \parallel M'L'$. We also have $C'D' \parallel CD$, $B'C' \parallel BC$, $A'B' \parallel AB$ hence $\triangle DCB \sim \triangle D'C'B'$ and $\triangle ABC \sim \triangle A'B'C'$. Thus there exists a homothety that takes $A'B'C'D'$ to $ABCD$ and this homothety will map $P'$ into the point $P$ with the required properties.

![Diagram](attachment:diagram.png)

Let $M, N, P, Q$ be the points of tangency of $\omega$ with $AB, BC, CD$, and $DA$, respectively. We have that $AB + AD = AB + AQ - QD = AB + AM - DP = BM - CP + CD = BN - CN + CD = BC + CD$. Denote by $X$ and $Y$ the points of tangency of $\omega_1$ and $\omega_2$ with $AC$. Then we have $AB = AX + BC - CX$ and $AD = AY + CD - CY$. Together with $AB + AD = BC + CD$ this yields to $AX - CX = CY - AY$. Since $AX + CX = CY + AY$ we conclude that $AX = CY$ hence $Y$ is the point of tangency of $AC$ and the excircle $\omega_2$ of $\triangle ABC$ that corresponds to $B$. Similarly, the excircle $\omega_2$ corresponding to $D$ of $\triangle ADC$ passes through $X$. Consider the homothety that maps $\omega_2$ to $\omega$. Denote by $Z$ the image of $Y$ under this homothety. $Z$ belongs to the tangent of $\omega$ that is parallel to $AC$. Therefore $Z$ is the image of $X$ under the homothety with center $D$ that maps $\omega_2$ to $\omega$. Denote by $X'$ and $Y'$ the intersections of $DX$ and $BY$ with $\omega_2$ and $\omega_1$ respectively. Circles $\omega_1$ and $\omega_2$ are homothetic with center $B$, hence $Y'$ the image of $Y$ under this homothety. Moreover, $Y'$ belongs to the tangent of $\omega_1$ that is parallel to $AC$. This implies that $XY'$ is a diameter of $\omega_1$. Similarly, $X'Y'$ is a diameter of $\omega_2$. This implies that $X'Y' \parallel XY'$ which means that $\triangle ZXY' \sim \triangle ZXY$ and $Z$ is the center of homothety that maps $\omega_2$ to $\omega_1$. This finishes the proof of the required statement.

21. Assume the contrary. If two of the numbers are the same then so are all three of them. Let us therefore assume that all of $a, b, c$ are different. The given conditions imply that

$$\frac{a^2 - b^n}{a - b} \cdot \frac{b^n - c^n}{b - c} \cdot \frac{c^n - a^n}{c - a} = -p^3,$$

which immediately implies that some of the numbers $a, b, c$ have to be negative. Moreover, $n$ can't be odd since otherwise each of the fractions would be positive. Assume first that $p$ is odd. Since $2 \mid a^2 - b^n = a^{n-1} + a^{n-2}b + \cdots + b^n$ the numbers $a$ and $b$ have to be of different parity. Similarly, $2 \mid b - c$ and $2 \mid c - a$ which is not possible.

We are left with the case $p = 2$. Writing $n = 2m$ we derive $(a^m + b^m) \cdot (b^m + c^m) \cdot (c^m + a^m) \cdot \frac{a^2 - b^n}{a - b} \cdot \frac{b^n - c^n}{b - c} \cdot \frac{c^n - a^n}{c - a} = -8$. This means that $a^m + b^m = \pm 2$, $a^m - b^m = \pm (a - b)$ and analogous equalities hold for the pairs $(b, c)$ and $(c, a)$. If $m$ is even then $|a| = |b| = |c| = 1$ which means that at least two of $a, b, c$ have to be the same.
If \( m \) is odd then \( \pm 2 = a^m + b^m \) is divisible by \( a + b \). Since \( a^m + b^m \equiv a + b \pmod{2} \) we conclude that \( a + b = \pm 2 \). Similarly \( b + c = \pm 2 \) and \( c + a = \pm 2 \). At least two of \( a, b, c \) have to be the same which is a contradiction.

**Remark.** The statement of the problem remains valid if we replace the assumption that \( p \) is prime with the assumption \( 2 \mid p \) or \( p = 2 \).

22. Assume the contrary. Without loss of generality we may assume that these numbers are relatively prime (otherwise we could divide them by their common divisor). We may also assume that \( a_1 < a_2 < \cdots < a_n \). For each \( i \in \{1, 2, \ldots, n-1\} \) there exists \( j \in \{1, 2, \ldots, n-1\} \) such that \( a_{n} + a_i \mid 3a_j \). This together with \( a_n + a_i > a_j \) implies that \( a_n + a_i \) is divisible by 3 for all \( i \).

There exists \( k \in \{1, 2\} \) such that \( a_n \equiv k \pmod{3} \) and \( a_1 \equiv 3 - k \pmod{3} \) for all \( i \neq n \). For each \( i \in \{1, 2, \ldots, n-2\} \) there exists \( j \) such that \( a_{n-1} + a_i \mid 3a_j \). Since \( a_{n-1} + a_i \) is not divisible by 3 we must have \( a_{n-1} + a_i \mid a_j \) hence \( j = n \) and we conclude that \( a_{n-1} + a_i \mid a_n \) for all \( i \in \{1, 2, \ldots, n-2\} \). Let \( l \in \{1, 2, \ldots, n\} \) be such an integer for which \( a_n + a_{n-1} \mid 3a_l \). Adding the inequalities \( a_n + a_{n-1} \leq 3a_l \) and \( a_n + a_i \leq a_l \) gives that \( a_{n-1} \leq a_l \) thus either \( l = n \) or \( l = n-1 \).

In the first case \( v(a_{n-1} + a_n) = 3a_n \) for some \( u \in \mathbb{N} \). We immediately see that \( u < 3 \) and \( u > 1 \). Hence \( u = 2 \) and \( 2a_{n-1} = a_n \). However, this is impossible since for each \( i \in \{1, 2, \ldots, n-2\} \) the number \( a_{n-1} + a_i \) divides \( a_n = 2a_{n-1} \).

On the other hand, if \( a_{n-1} + a_n \mid 3a_{n-1} \) then there exists \( v \in \mathbb{N} \) for which \( v(a_{n-1} + a_n) = 3a_{n-1} \). If \( v \geq 2 \) then \( 2a_{n-1} + 2a_n \leq 3a_{n-1} \) which is impossible. Hence \( v = 1 \) and we get \( a_n = 2a_{n-1} \). In the same way as in the previous case we get a contradiction.

23. We will use the induction on \( n \). Observe that \( a_n \geq (a_{n+1}, a_n) > a_{n-1} \). Obviously, \( a_0 \geq 1 \), and \( a_1 \geq a_0 + 1 \geq 2 \). From \( a_{k+1} - a_k \geq (a_{k+1}, a_k) \geq a_{k-1} + 1 \) we get \( a_2 \geq 4 \) and \( a_3 \geq 7 \). It is impossible to have \( a_3 = 7 \) since \( (a_3, a_2) = a_2 = 2 \) would imply \( a_2 = 7 = a_3 \). Hence we have that the statement is satisfied for \( n \in \{0, 1, 2, 3\} \).

Assume that \( n \geq 2 \) and \( a_i \geq 2^i \) for all \( i \in \{0, 1, \ldots, n\} \). We need to prove that \( a_{n+1} \geq 2^{n+1} \). Let us denote \( d_n = (a_{n+1}, a_n) \). We have \( d_n > a_{n-1} \). Let \( a_{n+1} = kd_n \) and \( a_n = ld_n \). If \( k \geq 4 \) we are done because \( a_{n+1} \geq 4d_n > 4a_{n-1} + 2 \cdot 2^{n-1} = 2^n + 1 \).

If \( l \geq 3 \) then \( a_{n+1} > a_n \) implies \( k \geq 4 \). If \( l = 1 \) then \( a_{n+1} \geq 2a_n \geq 2^{n+1} \). Hence the only remaining case to consider is \( a_n = 3d_n, a_{n-1} = 2d_n \). Obviously, \( d_{n-1} = (2d_n, a_{n-1}) > a_{n-2} \). If \( a_{n-1} = d_{n-1} \) then from \( a_{n-1} < d_n \) and \( a_{n-1} \mid 2d_n \) we get \( \frac{a_{n-1}}{a_{n-2}} \geq 3 \) and \( d_n \geq \frac{3}{2} a_{n-1} \geq \frac{3}{2} \cdot 2^{n-1} \). Now \( a_{n+1} = 3d_n \geq 9 \cdot 2^{n-2} > 2^n + 1 \).

If \( a_{n-1} > 3d_{n-1} \) then \( d_n > a_{n-1} \geq 3d_{n-1} \). Since \( d_{n-1} = (2d_n, a_{n-1}) \) there exists \( s \in \mathbb{N} \) such that \( 2d_n = sd_{n-1} \). This implies that \( d_n > 3 \cdot \frac{2^s}{s} \) which means \( s > 6 \), or \( s \geq 7 \). Therefore \( 2d_n \geq 7d_{n-1} > 7 \cdot 2^{n-2} \) and \( a_{n+1} = 3d_n > 2^{n+1} \).

It remains to consider the case \( a_{n-1} = 2d_{n-1} \). From \( (2d_n, 2d_{n-1}) = d_{n-1} \) we conclude that \( d_n = \frac{d_{n-1}}{w} \) for some odd integer \( w \geq 3 \). From \( a_{n-1} < d_n \) we get \( 2d_{n-1} < d_n \) hence \( w \geq 5 \). If \( w \geq 7 \) then \( a_{n+1} \geq 3 \cdot 7 \cdot \frac{d_{n-1}}{w} > 21 \cdot 2^{n-3} > 2^n + 1 \) hence it remains to consider the case \( w = 5 \). We now have \( 2^{n-3} \leq a_{n-3} < d_{n-2} = (2d_{n-1}, a_{n-2}) \). If \( a_{n-2} \geq 2d_{n-2} \) then \( 2d_{n-1} = 3d_{n-2} > 3 \cdot 2^{n-3} \). There-
24. First we prove that the numbers \( \left( \binom{2^n}{k} \right) \) are all odd. Let \( M \) be the largest integer for which \( 2^M \) divides \((2^n - 1)\). Then \( M = \sum_{i=1}^{n-1} \left[ 2^{n-i} - 1 \right] = \sum_{i=1}^{n-1} \left( 2^{n-i} - 1 \right) \).

The largest number \( N \) for which \( 2^N \) divides \( k! \cdot (2^n - 1 - k)! \) satisfies

\[
N = \sum_{i=1}^{n-1} \left( \left\lfloor \frac{k}{2^i} \right\rfloor + \left\lceil \frac{2^{n-i} - k + 1}{2^i} \right\rceil \right).
\]

Each summand on the right-hand side is equal to \( 2^{n-i} - 1 \) (write \( k = q_i \cdot 2^j + r_i \), for \( 0 \leq r_i < 2^j \)). Hence \( M = N \) and \( \left( \binom{2^n}{k} \right) \) is odd.

Let us prove that \( \left( \binom{2^n}{k} \right) \) give different remainders modulo \( 2^n \). This is valid for \( n = 1 \). Assume that this holds for some \( n > 1 \). We claim that the sets \( A_i = \left\{ \binom{2^{n+1}+1}{2i}, \binom{2^{n+1}+1}{2i+1} \right\} \) and \( B_i = \left\{ \binom{2^{n+1}+1}{2i+1}, 2^{n+1}+1 - \binom{2^{n+1}+1}{2i} \right\} \) are the same modulo \( 2^{n+1} \) for each \( i = 0, 1, \ldots, 2^n - 1 \). We also claim that all numbers from \( \bigcup_{i=0}^{2^n-1} B_i \) are different modulo \( 2^{n+1} \). These two claims will imply the desired result. Let us show that \( \binom{2^{n+1}}{2i} \equiv -\binom{2^{n+1}}{2i+1} \pmod{2^{n+1}} \) and that one of these two numbers is congruent to \( \binom{2^n}{i} \). The first congruence follows from

\[
\binom{2^{n+1}+1}{2i} = \binom{2^{n+1}+1}{2i+1} - \binom{2^{n+1}}{2i+1} = \frac{2^{n+1}}{2i+1} \left( \binom{2^{n+1}}{2i+1} - \binom{2^{n+1}}{2i} \right) = -\binom{2^{n+1}+1}{2i+1} \pmod{2^{n+1}},
\]

while the second is true because

\[
\binom{2^{n+1}+1}{2i} = \prod_{k=0}^{2^{n+1}-(2i+1)} \frac{2^{n+1}-(2k+1)}{2k+1} \prod_{k=1}^{2^{n+1}-2k} \frac{2^{n+1}-2k}{2k} = \prod_{k=0}^{\frac{2^{n+1}-1}{2k+1}} \frac{2^{n+1}-(2k+1)}{2k+1} \cdot \prod_{k=1}^{\frac{2^n-1}{k}} \frac{2^n-k}{k} \equiv (-1)^j \binom{2^n+1}{i} \pmod{2^{n+1}}.
\]

It remains to show that \( \bigcup_{i=0}^{2^n-1} B_i \) have all elements different modulo \( 2^{n+1} \).

Induction hypothesis implies that \( \binom{2^{n-1}}{i} \) has different remainder than \( \binom{2^{n-1}}{j} \) for \( i \neq j \). The same holds for \( 2^{n+1} - \binom{2^{n-1}}{i} \) and \( 2^{n+1} - \binom{2^{n-1}}{j} \). From \( \binom{2^{n-1}}{2k+1} + \binom{2^{n-1}}{2k} \equiv 2^n \) we have that \( \binom{2^{n}}{i} \equiv 2^{n+1} - \binom{2^{n-1}}{i} \pmod{2^{n+1}} \) if and only if there exists \( k \) such that \( \{i, j\} = \{2k, 2k+1\} \) for some \( k \). However, in that case \( \binom{2^{n-1}}{i} + \binom{2^{n-1}}{j} = \frac{2^{n}}{2k+1} \binom{2^{n}}{2k} \neq 0 \pmod{2^{n+1}} \).
25. If \( p \) is a prime number, then \( d(f(p)) \) has \( p \) divisors, and must be a power of a prime. Hence \( f(p) = q^{p-1} \) for some prime number \( q \). Let us show that \( q = p \).

Consider first the case \( p > 2 \). From \( f(2p) \mid (2 - 1) \cdot p^{2p-1} \cdot f(2) \) and \( f(2p) \mid (p - 1) \cdot 2^{2p-1} \cdot f(p) = (p - 1) \cdot 2p^{p-1} \cdot q^{p-1} \) we conclude that \( f(2p) \mid (p^{2p-1}, f(2), (p - 1) \cdot 2^{2p-1} \cdot q^{p-1}) \). Since \( f(2p) \) has \( 2p \) divisors and \( f(2) \) is prime this is a contradiction. We also have \( f(2) = 2 \). Indeed, this follows from \( f(6) \mid 3^{6-1} \cdot f(2), f(6) \mid 2 \cdot 2^{6-1} \cdot 3^{3-1} \), and \( d(f(6)) = 6 \).

Assume now that \( x = p_1^{a_1} \cdots p_n^{a_n} \) is a prime factorization of \( x \) with \( p_1 < \cdots < p_n \). Let \( f(x) = q_1^{b_1} \cdots q_m^{b_m} \). From \( d(f(x)) = p_1^{a_1} \cdots p_n^{a_n} = (b_1 + 1) \cdots (b_m + 1) \) we conclude that \( b_i \geq p_i - 1 \) for all \( i \). The relation \( f(x) \mid (p_1 - 1) \cdot (p_1^{a_1 - 1} \cdots p_n^{a_n})^{x-1} \cdot f(p_1) \) yields to \( q_1, \ldots, q_m \in \{p_1, \ldots, p_n\} \). Hence for each prime \( p \) and each \( a \in \mathbb{N} \) there is \( b \in \mathbb{N} \) such that \( f(p^a) = p^b \). From \( p^n = b + 1 \) we get \( f(p^n) = p^n \).

Now assume that \( x \in \mathbb{N} \). There are integers \( a_1, \ldots, a_n, b_1, \ldots, b_n \) such that \( x = p_1^{a_1} \cdots p_n^{a_n} \) and \( f(x) = p_1^{b_1} \cdots p_n^{b_n} \). For each \( i \in \{1, \ldots, n\} \) we have \( f(x) \mid (p_i^{a_i} - 1) \cdot (x/p_i^{a_i})^{x-1} \cdot p_i^{a_i - 1} \cdot p_i^{a_i - 2} = x \). Since \( f(x) = x \) we must have \( b_i = p_i^{a_i} - 1 \) for all \( i \) and \( f(x) = p_1^{a_1 - 1} \cdots p_n^{a_n - 1} \).

It is easy to verify that function \( f \) defined by the previous relation satisfies the required conditions.

26. If \( p \) is any prime number of the form \( p \equiv 1 \pmod{4} \) we know that \( \left( \frac{-1}{p} \right) = 1 \) and there are exactly two numbers \( n, m \in \{0, 1, 2, \ldots, p - 1\} \) whose square is congruent to \(-1\) modulo \( p \). Since the sum of these two numbers is equal to \( p \), one of them is smaller than \( p/2 \). Assuming that \( n < p/2 \) let us denote \( k = p - 2n \). It suffices to prove that there exist infinitely many prime numbers \( p \) for which \( k > \sqrt{2n} \). From \( p \mid n^2 + 1 = \frac{p^2 - 2pk + k^2}{4} + 1 \) we conclude that \( p \mid k^2 + 4 \). This implies that \( k^2 \geq p - 4 \). It suffices to prove that \( p - 4 > 2n \), i.e. \( 4 < p - 2n = k \) for infinitely many values of \( p \). However, this will be satisfied since \( k \geq \sqrt{p - 4} > 4 \) for \( p > 20 \), and there are infinitely many prime numbers greater than 20 that are congruent to 1 modulo 4.
A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $B(A, B, C), A - B - C$: indicates the relation of betweenness, i.e., that $B$ is between $A$ and $C$ (this automatically means that $A, B, C$ are different collinear points).
- $A = l_1 \cap l_2$: indicates that $A$ is the intersection point of the lines $l_1$ and $l_2$.
- $AB$: line through $A$ and $B$, segment $AB$, length of segment $AB$ (depending on context).
- $[AB]$: ray starting in $A$ and containing $B$.
- $(AB)$: ray starting in $A$ and containing $B$, but without the point $A$.
- $(AB)$: open interval $AB$, set of points between $A$ and $B$.
- $[AB]$: closed interval $AB$, segment $AB$, $(AB) \cup \{A, B\}$.
- $(AB)$: semiopen interval $AB$, closed at $B$ and open at $A$, $(AB) \cup \{B\}$.
The same bracket notation is applied to real numbers, e.g., $[a, b) = \{x \mid a \leq x < b\}$.
- $ABC$: plane determined by points $A, B, C$, triangle $ABC$ ($\triangle ABC$) (depending on context).
- $[AB, C]$: half-plane consisting of line $AB$ and all points in the plane on the same side of $AB$ as $C$.
- $(AB, C) [AB, C$ without the line $AB$.}
22 A Notation and Abbreviations

○ $\langle \vec{a}, \vec{b} \rangle$, $\vec{a} \cdot \vec{b}$: scalar product of $\vec{a}$ and $\vec{b}$.

○ $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle $ABC$ (unless otherwise indicated).

○ $k(O, r)$: circle $k$ with center $O$ and radius $r$.

○ $d(A, p)$: distance from point $A$ to line $p$.

○ $S_{A_1A_2...A_n}$, $[A_1A_2...A_n]$: area of $n$-gon $A_1A_2...A_n$ (special case for $n = 3$, $S_{ABC}$: area of $\triangle ABC$).

○ $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).

○ $\mathbb{Z}_n$: the ring of residues modulo $n$, $n \in \mathbb{N}$.

○ $\mathbb{Z}_p$: the field of residues modulo $p$, $p$ being prime.

○ $\mathbb{Z}[x], \mathbb{R}[x]$: the rings of polynomials in $x$ with integer and real coefficients respectively.

○ $R^*$: the set of nonzero elements of a ring $R$.

○ $R(\alpha)$, $R[\alpha]$, where $\alpha$ is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.

○ $X_0$: $X \cup \{0\}$ for $X$ such that $0 \notin X$.

○ $X^+, X^-, aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}, a, b \in \mathbb{R}$.

○ $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to $x$.

○ $\lceil x \rceil$: the smallest integer greater than or equal to $x$.

The following is notation simultaneously used in different concepts (depending on context).

○ $|AB|$, $|x|$, $|S|$: the distance between two points $AB$, the absolute value of the number $x$, the number of elements of the set $S$ (respectively).

○ $(x, y)$, $(m, n)$, $(a, b)$: (ordered) pair $x$ and $y$, the greatest common divisor of integers $m$ and $n$, the open interval between real numbers $a$ and $b$ (respectively).

### A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

○ w.l.o.g.: without loss of generality.

Other abbreviations include:

○ RHS: right-hand side (of a given equation).
- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.
B

## Codes of the Countries of Origin

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