LOWER BOUND FOR A FOURTH-ORDER DERIVATIVE OF FIRST-PASSAGE PERCOLATION WITH RESPECT TO THE ENVIRONMENT

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ABSTRACT. The variance in first-passage percolation can be bounded in terms of the L^2-norms of derivatives with respect to the environment. The study of higher-order derivatives in this context is still in its early stages, and only a few results are currently known. In this work, we prove that -2 is the optimal lower bound for the fourth-order derivative. To establish this result, we develop an algorithm and implement a computer program that constructs a rigorous mathematical proof of the inequality.

1. INTRODUCTION

1.1. **Model.** We will use the same model and the definitions as in [5]. We refer the reader to the same paper for more comprehensive historical remarks and overview of the literature. For completeness, we will define the model here and state the results that we will need for the proofs.

Let $a < b$ be two fixed positive real numbers, and let $p \in (0,1)$ be a fixed probability. We consider a finite subgraph of the integer lattice $\mathbb{Z}^d$ (for $d \geq 2$), restricted to the box $[-2n,2n]^d$. Two vertices $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)$ are connected by an edge if they are nearest neighbors, i.e., if $|x_1-y_1|+\cdots+|x_d-y_d| = 1$.

Each edge e in this graph is independently assigned a random passage time, taking the value a with probability p and b with probability $1-p$. Let W_n denote the set of all such edges, and define the sample space as $\Omega_n = \{a,b\}^{W_n}$, where each element $\omega \in \Omega_n$ specifies a particular realization of passage times on the edges.

Given a configuration ω and a path γ (a sequence of adjacent edges), the passage time $T(\gamma,\omega)$ is defined as the sum of the passage times assigned to the edges in γ. For any two vertices u and v, the function $f(u,v,\omega)$ denotes the minimal passage time over all paths connecting u to v in the configuration ω.

When the destination vertex v is fixed, we may write $f_n(\omega)$, or simply f_n, to refer to $f(0,nv,\omega)$.

A path γ is called *geodesic* if the minimum $f_n(\omega)$ is attained at γ, i.e. if $f_n(\omega) = T(\gamma,\omega)$.

1.2. **Environment derivatives.** If we denote by W_n the set of all edges, then the sample space is $\Omega_n = \{a,b\}^{W_n}$. We will often omit the subscript n, when there is no danger of confusion. For each edge j and each $\omega \in \Omega$, we define $\sigma_j^a(\omega)$ as the element of Ω whose j-th coordinate is changed from ω_j to a, regardless of what the original value ω_j was. The operation σ_j^b is defined in analogous way. Formally, for $\delta \in \{a,b\}$, we define $\sigma_j^\delta : \Omega \to \Omega$ with

$$\left[\sigma_j^\delta(\omega)\right]_k = \begin{cases} \omega_k, & k \neq j, \\ \delta, & k = j. \end{cases} \tag{1}$$

If $\varphi : \Omega \to \mathbb{R}$ is any random variable, then the *first order environment derivative* $\partial_j \varphi$ is the random variable defined as

$$\partial_j \varphi \quad = \quad \varphi \circ \sigma_j^b - \varphi \circ \sigma_j^a. \tag{2}$$

For two distinct vertices k and l, we will give the name *second order environment derivative* to the quantity $\partial_k \partial_l \varphi$. In general, if S is a non-empty subset of W, the operator $\partial_S \varphi$ is defined recursively as

$$\partial_S \varphi \quad = \quad \partial_{S\setminus\{j\}} \left(\partial_j \varphi\right), \tag{3}$$

where j is an arbitrary element of S. The definition (3) is independent on the choice of j, since a simple induction can be used to prove that for $S = \{s_1, \dots, s_m\}$, the following holds

$$\partial_S \varphi \quad = \quad \sum_{\theta_1 \in \{a,b\}} \cdots \sum_{\theta_m \in \{a,b\}} (-1)^{\mathbf{1}_a(\theta_1)+\cdots+\mathbf{1}_a(\theta_m)} \varphi \circ \sigma_{s_1}^{\theta_1} \circ \cdots \circ \sigma_{s_m}^{\theta_m}. \tag{4}$$

The function $\mathbf{1}_a : \{a,b\} \to \{0,1\}$ in (4) assigns the value 1 to a and 0 to b.

1.3. **Bounds on variance.** The following result from [5] provides a bound on the variance in terms of environment derivatives which generalizes Talagrand's theorem from [7].

Theorem 1. *Let f be a random variable on Ω. For every integer $k \geq 1$, there exists a real constant C and an integer n_0 such that for $n \geq n_0$, the following inequality holds*

$$\begin{aligned} var(f) \quad \leq \quad & \sum_{M \subseteq W, 1 \leq |M| < k} (p(1-p))^{|M|} \left(\mathbb{E}\left[\partial_M f\right]\right)^2 \\ & + C \cdot \sum_{M \subseteq W, |M| = k, \|\partial_M f\|_1 \neq 0} \frac{\|\partial_M f\|_2^2}{1 + \left(\log \frac{\|\partial_M f\|_2}{\|\partial_M f\|_1}\right)^k}, \end{aligned} \tag{5}$$

where $\|g\|_p$ is the L^p-norm of the function g defined as

$$\|g\|_p = \left(\int_\Omega |g|^p \, d\mathbb{P}\right)^{1/p} = \left(\mathbb{E}[|g|^p]\right)^{1/p}.$$

A widely conjectured upper bound for the variance in first-passage percolation is $C \cdot n^{2\chi}$, where the exponent χ depends on the dimension d. In two dimensions, current heuristics and numerical evidence suggest that $\chi = \frac{1}{3}$ [1]. A rigorous upper bound of $\chi \leq \frac{1}{2}$ was proven by Kesten in 1993 [4], but a formal proof that χ is strictly less than $\frac{1}{2}$ remains elusive. For dimensions $d > 2$, even conjectural values of χ are not firmly established. Nevertheless, as noted in [1], it is widely believed that χ remains positive in all dimensions, with its value tending to zero as the dimension increases.

The best known upper bound on the variance is currently $C \cdot \frac{n}{\log n}$, due to the work of Benjamini, Kalai, and Schramm [2]. Their argument relies on Talagrand's inequality [7], which involves first-order environment derivatives.

We conjecture that the L^2-norms $|\partial_M f|_2$ are small, particularly in dimensions $d \geq 3$, where exponential decay may occur. However, at present, these quantities remain analytically difficult to control.

A thorough understanding of environment derivatives would lead to a complete understanding of the variance. As shown in [5], the variance can be decomposed as

$$\mathrm{var}(f) \quad = \quad \sum_{M \subseteq W, M \neq \emptyset} (p(1-p))^{|M|} \left(\mathbb{E}\left[\partial_M f\right]\right)^2,$$

which highlights the central role of the L^2-norms of the derivatives $\partial_M f$ in determining fluctuation behavior. Despite their importance, general bounds for these norms are not yet available.

This paper contributes to the analysis of environment derivatives by completing the picture of almost sure bounds for all orders up to four. Using algebraic results from [5], we construct a computer-assisted proof showing that -2 is a lower bound for the fourth-order derivative [6].

1.4. **Almost sure bounds on environment derivatives.** The sequence $(\mathscr{U}_1, \mathscr{U}_2, \dots)$ represents the most optimal upper bounds for environment derivatives. The number $\mathscr{U}_k$ is defined as the best upper bound on the k-th order environment derivative, i.e.

$$\mathscr{U}_k = \frac{1}{b-a} \max \left\{ \partial_S f_n(\omega) : n \in \mathbb{N}, S \subseteq W_n, |S| = k, \omega \in \Omega_n \right\}. \tag{6}$$

The sequence $(\mathscr{L}_1, \mathscr{L}_2, \dots)$ of the most optimal lower bounds is defined in an analogous way

$$\mathscr{L}_k = \frac{1}{b-a} \min \left\{ \partial_S f_n(\omega) : n \in \mathbb{N}, S \subseteq W_n, |S| = k, \omega \in \Omega_n \right\}. \tag{7}$$

The re-scaling factor $b-a$ is included to make the numbers $\mathscr{U}_k$ and $\mathscr{L}_k$ constant and independent on a and b.

Theorem 2. *The first four values of $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ are given in the table below.*

k	1	2	3	4
$\mathscr{U}_k$	1	1	2	3
$\mathscr{L}_k$	0	-1	-1	-2

(8)

Theorem 3. *The sequences $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ satisfy*

$$(\forall k \geq 1) \qquad \mathscr{U}_{k+1} \leq \mathscr{U}_k - \mathscr{L}_k \quad \text{and} \quad \mathscr{L}_{k+1} \geq \mathscr{L}_k - \mathscr{U}_k; \tag{9}$$

$$(\forall k \geq 2) \qquad \mathscr{U}_k \leq 2^{k-2} \quad \text{and} \quad |\mathscr{L}_k| \leq 2^{k-2}; \tag{10}$$

$$(\exists k_0)(\forall k \geq k_0) \qquad \mathscr{U}_k \geq \sqrt[4]{3}^k \quad \text{and} \quad |\mathscr{L}_k| \geq \sqrt[4]{3}^k. \tag{11}$$

Except for $\mathscr{L}_4 \geq -2$, all other assertions of Theorems 2 and 3 were proved in [5].

In this paper we will prove $\mathscr{L}_4 \geq -2$. Our main result was obtained through a computer-assisted proof [6]. The problem was first reduced to 12 cases based on the structure of the environment. In each case, the derivative was expressed as a sum of 16 terms. The computer program then reorganized these terms and grouped them into pairs. For each arrangement, the program tested possible bounds for each pair. Eventually, it succeeded in finding configurations where significant cancellations occurred, leading to the bound $\mathscr{L}_4 \geq -2$.

In this paper, we describe the algorithm and present the proof of the inequality as obtained by the program.

2. Essential and influential edges

Except for Proposition 1 below, the results in this section were proved in [5]. Proposition 1 is trivial, but so important that it must be listed.

Proposition 1. *For every $i \neq j$, every $\alpha, \beta \in \{a,b\}$, and every random variable φ,*

$$\begin{aligned} \sigma_i^\alpha \circ \sigma_i^\beta &= \sigma_i^\alpha; && (12)\\ \sigma_i^\alpha \circ \sigma_j^\beta &= \sigma_j^\beta \circ \sigma_i^\alpha; && (13)\\ (\partial_i \varphi) \circ \sigma_i^\alpha &= \partial_i \varphi; && (14)\\ \partial_i \partial_i \varphi &= 0; && (15)\\ \partial_i \partial_j \varphi &= \partial_j \partial_i \varphi; && (16)\\ \varphi \cdot 1_{\omega_i = \alpha} &= \varphi \circ \sigma_i^\alpha \cdot 1_{\omega_i = \alpha}. && (17) \end{aligned}$$

For a fixed edge $j \in W_n$, define the events A_j and $\hat{A}_j$ with

$$\begin{aligned} A_j &= \left\{ \omega \in \Omega_n : \partial_j f(\omega) \neq 0 \right\}; && (18)\\ \hat{A}_j &= \left\{ \omega \in \Omega_n : \partial_j f(\omega) = b - a \right\}. && (19) \end{aligned}$$

The edge j is called *influential* if the event A_j occurred, and *very influential* if $\hat{A}_j$ occurred. The edge j is called *essential* if each geodesics passes through j, and *semi-essential* if at least one geodesic passes through j. We will denote by E_j the event that the edge j is essential and by $\hat{E}_j$ the event that the edge j is semi-essential.

Proposition 2. *The events E_j, A_j, $\hat{E}_j$, and $\hat{A}_j$ satisfy*

$$\begin{aligned} A_j &= (\sigma_j^a)^{-1}(E_j); && (20)\\ \hat{A}_j &= (\sigma_j^b)^{-1}(\hat{E}_j). && (21) \end{aligned}$$

Proposition 3. *The following two propositions hold for every $\omega \in \Omega$.*

(a) *If $\sigma_j^a(\omega) \in E_j^C$, then $f(\sigma_j^b(\omega)) = f(\sigma_j^a(\omega))$;*
(b) *If $\sigma_j^b(\omega) \in \hat{E}_j$, then $f(\sigma_j^b(\omega)) = f(\sigma_j^a(\omega)) + (b-a)$.*

The sets $\{\omega_j = a\}$ and $\{\omega_j = b\}$ are the ranges of the transformations σ_j^a and σ_j^b, i.e.

$$\sigma_j^a(\Omega) = \{\omega_j = a\} \quad \text{and} \quad \sigma_j^b(\Omega) = \{\omega_j = b\}. \tag{22}$$

Proposition 4. *For every $j \in W$, the events E_j, $\hat{E}_j$, A_j, and $\hat{A}_j$ satisfy*

$$\sigma_j^a(A_j) = \sigma_j^a(A_j \cap \{\omega_j = b\}) = A_j \cap \left\{\omega_j = a\right\} = E_j \cap \left\{\omega_j = a\right\}; \tag{23}$$

$$\sigma_j^b(\hat{A}_j) = \sigma_j^b(\hat{A}_j \cap \{\omega_j = a\}) = \hat{A}_j \cap \left\{\omega_j = b\right\} = \hat{E}_j \cap \left\{\omega_j = b\right\}. \tag{24}$$

Proposition 5. *For every $j \in W$, the events E_j, $\hat{E}_j$, A_j, and $\hat{A}_j$ satisfy*

$$\begin{aligned} &E_j \subseteq \hat{E}_j, \quad \hat{A}_j \subseteq A_j,\\ &E_j \subseteq A_j, \quad \hat{A}_j \subseteq \hat{E}_j. && (25) \end{aligned}$$

Proposition 6. *Assume that $\omega \in E_j$. A path γ is a geodesic on ω if and only if it is a geodesic on $\sigma_j^a(\omega)$.*

If $\overrightarrow{\alpha} \in \{a,b\}^m$ and $\overrightarrow{v} \in W^m$, define $\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}} : \Omega \to \Omega$ as

$$\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}} = \sigma_{v_1}^{\alpha_1} \circ \cdots \circ \sigma_{v_m}^{\alpha_m}, \tag{26}$$

where $\alpha_1, \ldots, \alpha_m$ are the components of $\overrightarrow{\alpha}$ and $v_1, \ldots, v_m$ are the components of $\overrightarrow{v}$.

3. Results from graph theory that are used in proofs

3.1. **Helpful pairs.** Let us analyze the representation (4) for $\partial_S f$ for set S with m elements. We can partition all vectors from $\{a,b\}^m$ into two sets: V_m^+ and V_m^-. The vector belongs to V_m^+ if it has an even number of components a; otherwise the vector is in V_m^-. The environment derivative can be written as

$$\partial_S f(\omega) \quad = \quad \sum_{\overrightarrow{v}\in V_m^+} f(\sigma_S^{\overrightarrow{v}}(\omega)) - \sum_{\overrightarrow{v}\in V_m^-} f(\sigma_S^{\overrightarrow{v}}(\omega)). \tag{27}$$

Denote $\mathscr{V}_m = V_m^+ \times V_m^-$. We will say that $(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m$ is an *associated pair* if $\overrightarrow{u}$ and $\overrightarrow{w}$ differ in exactly one coordinate. Every associated pair $(\overrightarrow{u},\overrightarrow{w})$ satisfies

$$f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) \quad \in \quad [0, b-a]. \tag{28}$$

Let us partition the set V_m^+ as $V_m^+ = \bigcup_{k=0}^{\lfloor \frac{n}{2}\rfloor} V_m^+(2k)$, where $V_m^+(2k)$ is the subset of those vectors from V_m^+ that contain exactly $2k$ elements equal to a. We partition the set V_m^- as $V_m^- = \bigcup_{k=0}^{\lfloor \frac{n}{2}\rfloor} V_m^-(2k+1)$ in an analogous way. Let us partition $\mathscr{V}_m$ into two subsets $\mathscr{V}_m^{\uparrow}$ and $\mathscr{V}_m^{\downarrow}$ defined as

$$\mathscr{V}_m^{\uparrow} \quad = \quad \left\{(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m : (\exists i)\overrightarrow{u} \in V_m^+(2i),\ \overrightarrow{w} \in V_m^-(2i+1)\right\}, \tag{29}$$

$$\mathscr{V}_m^{\downarrow} \quad = \quad \left\{(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m : (\exists i)\overrightarrow{u} \in V_m^+(2i),\ \overrightarrow{w} \in V_m^-(2i-1)\right\}. \tag{30}$$

Observe that the following inequalities hold for every $\omega \in \Omega$

$$(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m^{\uparrow} \quad \Longrightarrow \quad f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) \in [0, b-a], \tag{31}$$

$$(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m^{\downarrow} \quad \Longrightarrow \quad f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) \in [-(b-a), 0]. \tag{32}$$

For fixed $S \subseteq W$ and fixed $\omega \in \Omega$, we define the subsets $\mathscr{H}_-^{\uparrow}(\omega)$, $\mathscr{H}_-^{\downarrow}(\omega)$, $\mathscr{H}_-(\omega)$, $\mathscr{H}_+^{\uparrow}(\omega)$, $\mathscr{H}_+^{\downarrow}(\omega)$, and $\mathscr{H}_+(\omega)$ of $\mathscr{V}_m$ in the following way

$$\mathscr{H}_-^{\uparrow}(\omega) \quad = \quad \left\{(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m^{\uparrow} : f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) = b-a\right\}; \tag{33}$$

$$\mathscr{H}_-^{\downarrow}(\omega) \quad = \quad \left\{(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m^{\downarrow} : f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) = 0\right\}; \tag{34}$$

$$\mathscr{H}_+^{\uparrow}(\omega) \quad = \quad \left\{(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m^{\uparrow} : f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) = 0\right\}; \tag{35}$$

$$\mathscr{H}_+^{\downarrow}(\omega) \quad = \quad \left\{(\overrightarrow{u},\overrightarrow{w}) \in \mathscr{V}_m^{\downarrow} : f(\sigma_S^{\overrightarrow{u}}(\omega)) - f(\sigma_S^{\overrightarrow{w}}(\omega)) = -(b-a)\right\}; \tag{36}$$

$$\mathscr{H}_-(\omega) \quad = \quad \mathscr{H}_-^{\uparrow}(\omega) \cup \mathscr{H}_-^{\downarrow}(\omega); \tag{37}$$

$$\mathscr{H}_+(\omega) \quad = \quad \mathscr{H}_+^{\uparrow}(\omega) \cup \mathscr{H}_+^{\downarrow}(\omega). \tag{38}$$

We say that the associated pair $(\overrightarrow{u},\overrightarrow{w})$ is a *negatively helpful pair* if it belongs to $\mathscr{H}_-(\omega)$. The associated pair $(\overrightarrow{u},\overrightarrow{w})$ is a *positively helpful pair* if it belongs to $\mathscr{H}_+(\omega)$.

For fixed S and ω, we define $H_+(S,\omega)$ as the largest number of disjoint positively helpful pairs. Similarly, we define $H_-(S,\omega)$ as the largest number of disjoint negatively helpful pairs.

3.2. **Matching tolerance.** Let us first review our definition of the graph $\mathscr{V}_m$. The vertices are the elements of $\{a,b\}^m$. For the purposes of this subsection it is most convenient to think of $\{a,b\}^m$ as the m-dimensional hypercube. Edges are drawn between associated pairs of vertices, i.e. between the m-tuples that differ in exactly one coordinate. This graph is bipartite and the partition of the vertices is the pair (V_m^+, V_m^-), where V_m^+ are those m-tuples from $\{a,b\}^m$ with even number of as and V_m^- are the m-tuples with odd number of

as. Let us recall that the *perfect matching* in the bipartite graph is defined as the partition of the vertex set $V_m^+ \cup V_m^-$ in two-element subsets such that the elements in each of the subsets are connected by an edge. In some literature, the term *marriage* or *Hall's marriage* is used for perfect matching.

We define the *matching tolerance* of the graph as the largest integer T for which the following holds: For every $R \leq T$ and every choice of R disjoint edges $(\overrightarrow{u_1}, \overrightarrow{w_1})$, ..., $(\overrightarrow{u_R}, \overrightarrow{w_R})$ from $\mathscr{V}_m$, after the removal of $\{\overrightarrow{u_1}, \dots, \overrightarrow{u_R}\}$ from V_m^+ and the removal of $\{\overrightarrow{w_1}, \dots, \overrightarrow{w_R}\}$ from V_m^-, there still exists a perfect matching on the remaining graph.

Let us denote by $T(\mathscr{V}_m)$ the matching tolerance of the graph $\mathscr{V}_m$. The graph $\mathscr{V}_m$ is isomorphic to C_2^m, where C_2 is a cycle of length 2. The matching tolerance has a simple relation to the matching preclusion $\mathrm{mp}(\mathscr{V}_m)$, defined as the minimal number of edges that has to be removed to make it impossible to find the perfect matching. The relationship is $T(\mathscr{V}_m) = \mathrm{mp}(\mathscr{V}_m) - 1$. We will need the following result, which is equivalent to Theorem 12 from [3].

Theorem 4. *For $m \geq 3$, the matching tolerance of $\mathscr{V}_m$ is $m-1$, i.e. $T(\mathscr{V}_m) = m-1$.*

Proposition 7. *Assume that S is a subset of W with m elements. The following inequalities hold*

$$\partial_S f(\omega) \geq -(2^{m-2} - \min\{H_-(S,\omega), m-1\})(b-a), \quad (39)$$

$$\partial_S f(\omega) \leq (2^{m-2} - \min\{H_+(S,\omega), m-1\})(b-a). \quad (40)$$

Proof. We will only prove (39). The proof of (40) is analogous. In order to prove (39), it suffices to prove that for every $k \leq T(\mathscr{V}_m) = m-1$, the existence of k disjoint negatively helpful pairs on ω implies

$$\partial_S f(\omega) \geq -(2^{m-2} - k)(b-a). \quad (41)$$

Assume that $\mathscr{A}$ is a subset of $\mathscr{V}_m$ and that each element of $\mathscr{A}$ is an associated pair (i.e. an edge in the graph). The set $\mathscr{A}$ can be partitioned into $\mathscr{A}^\uparrow$ and $\mathscr{A}^\downarrow$ in the following way

$$\mathscr{A}^\uparrow = \mathscr{A} \cap \mathscr{V}_m^\uparrow \quad \text{and} \quad \mathscr{A}^\downarrow = \mathscr{A} \cap \mathscr{V}_m^\downarrow.$$

Let $\mathscr{P}$ be the set of k disjoint negatively helpful pairs on the outcome ω. Let $\mathscr{M}$ be the perfect matching on the remaining bipartite graph that is obtained when all the components of $\mathscr{P}$ are removed from $V_m^+ \cup V_m^-$. For each $\mathscr{A} \subseteq \mathscr{V}_m$, we define $\Sigma_\partial(\mathscr{A})$ as

$$\Sigma_\partial(\mathscr{A}) = \sum_{(\overrightarrow{u}, \overrightarrow{w}) \in \mathscr{A}} \left(f(\sigma_S^{\overrightarrow{u}}(\omega) - f(\sigma_S^{\overrightarrow{w}}(\omega)) \right).$$

The equation (27) implies that the environment derivative $\partial_S f(\omega)$ satisfies

$$\partial_S f(\omega) = \Sigma_\partial(\mathscr{P}^\uparrow) + \Sigma_\partial(\mathscr{P}^\downarrow) + \Sigma_\partial(\mathscr{M}^\uparrow) + \Sigma_\partial(\mathscr{M}^\downarrow). \quad (42)$$

From the definition of negatively helpful pairs we have

$$\Sigma_\partial(\mathscr{P}^\uparrow) = (b-a)\left|\mathscr{P}^\uparrow\right| \quad \text{and} \quad \Sigma_\partial(\mathscr{P}^\downarrow) = 0. \quad (43)$$

On the other hand, we have $f(\sigma^{\overrightarrow{u}}(\omega)) - f(\sigma^{\overrightarrow{w}}(\omega)) \geq 0$ whenever $(\overrightarrow{u}, \overrightarrow{w}) \in \mathscr{M}^\uparrow$. If $(\overrightarrow{u}, \overrightarrow{w}) \in \mathscr{M}^\downarrow$, then $f(\sigma^{\overrightarrow{u}}(\omega)) - f(\sigma^{\overrightarrow{w}}(\omega)) \geq -(b-a)$. This lower bound was trivial, but on $\mathscr{M}^\downarrow$ we cannot count on any sharper bound. These last two bounds imply

$$\Sigma_\partial(\mathscr{M}^\uparrow) \geq 0 \quad \text{and} \quad \Sigma_\partial(\mathscr{M}^\downarrow) \geq -(b-a)\left|\mathscr{M}^\downarrow\right|. \quad (44)$$

We now use (42), (43), and (44) to obtain

$$\partial_S f(\omega) \quad \geq \quad (b-a) \cdot \left(\left| \mathscr{P}^{\uparrow} \right| - \left| \mathscr{M}^{\downarrow} \right| \right). \tag{45}$$

Observe that $\left|\mathscr{M}^{\uparrow}\right| + \left|\mathscr{P}^{\uparrow}\right| = 2^{m-2}$ and $\left|\mathscr{M}^{\downarrow}\right| + \left|\mathscr{P}^{\downarrow}\right| = 2^{m-2}$. Therefore,

$$\left|\mathscr{M}^{\downarrow}\right| \quad = \quad 2^{m-2} - \left|\mathscr{P}^{\downarrow}\right| = 2^{m-2} - \left(k - \left|\mathscr{P}^{\uparrow}\right|\right) = \left|\mathscr{P}^{\uparrow}\right| - k + 2^{m-2}.$$

The last equation implies that $\left|\mathscr{P}^{\uparrow}\right| - \left|\mathscr{M}^{\downarrow}\right| = k - 2^{m-2}$. The inequality (45) now turns into the desired inequality (41). As we discussed earlier, the inequality (39) follows from (41). □

3.3. **Case $m = 4$.** Since we know that $T(\mathscr{V}_4) = 3$, we can restate the Proposition 7 for the case $m = 4$ in a more convenient form.

Proposition 8. *Assume that $|S| = 4$. If there are 2 negatively helpful pairs on ω, then $\partial_S f(\omega) \geq -2(b-a)$.*

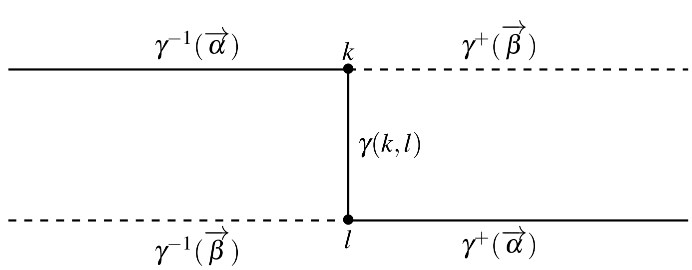
Proof. This follows directly from (41) by taking $k = 2$ and observing that this choice of k satisfies $k \leq T(\mathscr{V}_4)$. □

Hence, we only need to consider the cases in which the number of negatively helpful pairs is 0 or 1. We will write $S = \{v_1, v_2, v_3, v_4\}$ and we will write E_i and $\hat{E}_i$ instead of E_{v_i} and $\hat{E}_{v_i}$. We will also write $\sigma^{\overrightarrow{r}}(\omega)$ instead of $\sigma^{\overrightarrow{r}}_{(v_1,v_2,v_3,v_4)}(\omega)$.

3.4. **Direction of the flow.** The following proposition states that the direction of the flow cannot change if only few edges change their passage times.

Proposition 9. *Assume that k and l are two fixed edges and that $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ are two fixed vectors in $\{a,b\}^2$. Assume that $\gamma(\overrightarrow{\alpha})$ is a geodesic on $\sigma^{\overrightarrow{\alpha}}_{(k,l)}(\omega)$ that contains both k and l. Assume that $\gamma(\overrightarrow{\beta})$ is a geodesic on $\sigma^{\overrightarrow{\beta}}_{(k,l)}(\omega)$ that contains both k and l. Then, if k appears before l on the curve $\gamma(\overrightarrow{\alpha})$, then k must appear before l on the curve $\gamma(\overrightarrow{\beta})$.*

Proof. Assume the contrary, that on $\gamma(\overrightarrow{\beta})$ the edge l appears before k. Although the direction between k and l is different on the curves $\gamma(\overrightarrow{\alpha})$ and $\gamma(\overrightarrow{\beta})$, the passage times must be equal. We may assume that the two curves coincide between k and l.



Let us denote by $\gamma(k,l)$ this common path. Let $\gamma^-(\overrightarrow{\alpha})$ be the section of $\gamma(\overrightarrow{\alpha})$ before the edge k. Let $\gamma^+(\overrightarrow{\alpha})$ be the section of $\gamma(\overrightarrow{\alpha})$ after l. Define $\gamma^-(\overrightarrow{\beta})$ as the section of $\gamma(\overrightarrow{\beta})$ before the edge l and by $\gamma^+(\overrightarrow{\beta})$ be the section of $\gamma(\overrightarrow{\beta})$ after k. The curve γ_1 obtained by connecting $\gamma^-(\overrightarrow{\alpha})$, k, and $\gamma^+(\overrightarrow{\beta})$ may or may not be the geodesic on $\sigma^{\overrightarrow{\alpha}}(\omega)$. Hence, the passage time over $\gamma^-(\overrightarrow{\alpha})$, k, $\gamma(k,l)$, l, $\gamma^+(\overrightarrow{\alpha})$ (which is the same as $\gamma(\overrightarrow{\alpha})$) must be smaller than or equal than the passage time over the curve γ_1. We obtain

$$\begin{aligned} & T(\gamma^-(\overrightarrow{\alpha}))+\alpha_k+T(\gamma(k,l))+\alpha_l+T(\gamma^+(\overrightarrow{\alpha})) \\ \leq \quad & T(\gamma^-(\overrightarrow{\alpha}))+\alpha_k+T(\gamma^+(\overrightarrow{\beta})). \end{aligned} \tag{46}$$

In an analogous way we obtain

$$\begin{aligned} & T(\gamma^-(\overrightarrow{\beta}))+\beta_l+T(\gamma(k,l))+\beta_k+T(\gamma^+(\overrightarrow{\beta})) \\ \leq \quad & T(\gamma^-(\overrightarrow{\beta}))+\beta_l+T(\gamma^+(\overrightarrow{\alpha})). \end{aligned} \tag{47}$$

If we add the inequalities (46) and (47) and cancel the common terms, we obtain $\alpha_l+\beta_k+2T(\gamma(k,l))\leq 0$. The last inequality is not possible because the number $T(\gamma(k,l))$ is non-negative and each of α_l and β_l is strictly positive. □

4. Proof of $\mathscr{L}_4\geq -2$ in certain special cases

4.1. **Case in which a geodesic on $\sigma^{(a,a,a,a)}(\omega)$ omits some of the elements of S.** The next three propositions prove that the inequality $\partial_S f(\omega)\geq -2(b-a)$ holds unless each geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$ passes through all of the elements of S. In other words, the next three propositions will imply

$$\sigma^{(a,a,a,a)}(\omega)\notin E_1\cap E_2\cap E_3\cap E_4 \quad\Longrightarrow\quad \partial_S f(\omega)\geq -2(b-a). \tag{48}$$

Proposition 10. *Assume that there is a geodesic on $\sigma^{(a,a,a,a)}(\omega)$ that omits three elements of S. Then $\partial_S f(\omega)\geq -2(b-a)$.*

Proof. Assume that there were a geodesic on $\sigma^{(a,a,a,a)}$ that omits v_2, v_3, and v_4. We immediately have $f(\sigma^{(a,a,a,a)}(\omega))=f(\sigma^{(a,a,a,b)}(\omega))=f(\sigma^{(a,a,b,b)}(\omega))=f(\sigma^{(a,b,b,b)}(\omega))$. The pairs $((a,a,a,a),(a,a,a,b))$ and $((a,a,b,b),(a,b,b,b))$ would be two disjoint negatively helpful pairs, and the Proposition 8 would imply that $\partial_S f(\omega)$ is bounded below by $-2(b-a)$. □

Proposition 11. *Assume that there is a geodesic on $\sigma^{(a,a,a,a)}(\omega)$ that omits two elements of S. Then $\partial_S f(\omega)\geq -2(b-a)$.*

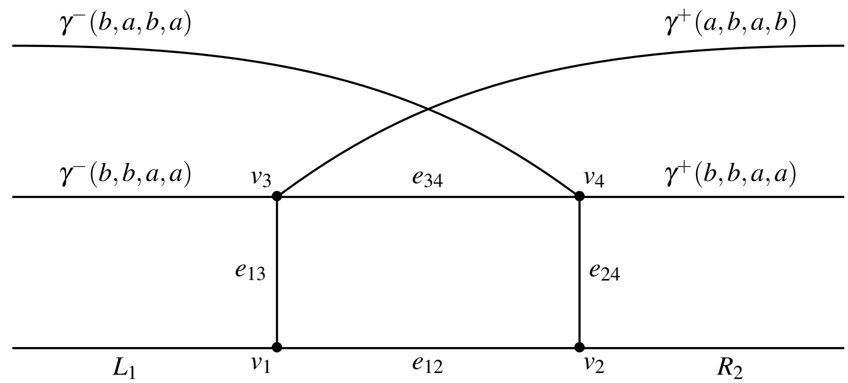
Proof. Due to the Proposition 10 we may assume that every geodesic on $\sigma^{(a,a,a,a)}(\omega)$ passes through at least two elements of S. Assume that the geodesic $\gamma(a,a,a,a)$ passes through v_1 and v_2 and omits v_3 and v_4. We may assume that v_1 appears before v_2 on $\gamma(a,a,a,a)$.

Each of the pairs $((a,a,a,a),(a,a,a,b))$ and $((a,a,a,a),(a,a,b,a))$ is negatively helpful. However, they are not disjoint, so we don't have a direct help from Proposition 8. However, let us look at the following 8 relations.

$$\begin{aligned} &\sigma^{(a,a,b,b)}(\omega)\in E_1\cap E_2\cap \hat{E}_3^C\cap \hat{E}_4^C; \quad &&\sigma^{(a,b,a,b)}(\omega)\in E_1\cap \hat{E}_2^C\cap E_3\cap \hat{E}_4^C;\\ &\sigma^{(a,b,b,a)}(\omega)\in E_1\cap \hat{E}_2^C\cap \hat{E}_3^C\cap E_4; \quad &&\sigma^{(b,a,a,b)}(\omega)\in \hat{E}_1^C\cap E_2\cap E_3\cap \hat{E}_4^C;\\ &\sigma^{(b,a,b,a)}(\omega)\in \hat{E}_1^C\cap E_2\cap \hat{E}_3^C\cap E_4; \quad &&\sigma^{(b,b,a,a)}(\omega)\in \hat{E}_1^C\cap \hat{E}_2^C\cap E_3\cap E_4. \end{aligned}$$

Observe that if any of the above 8 relations fails, then there is another negatively helpful pair, due to Proposition 3. One of its components is from $\mathscr{V}_4^+(2)$. This component is certainly not among (a,a,a,a), (a,a,a,b), (a,a,b,a). The other component is from $\mathscr{V}_m^-(3)$ or $\mathscr{V}_m^-(1)$. This other component can't be equal to both of (a,a,a,b) and (a,a,b,a); thus this new negatively helpful pair would be disjoint with one of the pairs $((a,a,a,a),(a,a,a,b))$ and $((a,a,a,a),(a,a,b,a))$. This would immediately imply $\partial_S f(\omega)\geq -2(b-a)$.

Assume now that $\gamma(a,a,a,a)$ passes through v_1 and v_2; omits v_3 and v_4; and that each of the 8 relations from above is satisfied. Let us consider a geodesic $\gamma(b,b,a,a)$ on $\sigma^{(b,b,a,a)}(\omega)$. This geodesic must go through v_3 and v_4 and must omit v_1 and v_2. Without loss of generality, we may assume that v_3 appears before v_4. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1 and $L_1 = T(\gamma^-(a,a,a,a))$. Let $\gamma^-(b,b,a,a)$ be the section of $\gamma(b,b,a,a)$ before v_3. Let $\gamma^+(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ after v_2 and $R_2 = T(\gamma^+(a,a,a,a))$. Let $\gamma^+(b,b,a,a)$ the section of $\gamma(b,b,a,a)$ after v_4.



Let us consider a geodesic $\gamma(a,b,a,b)$ on $\sigma^{(a,b,a,b)}(\omega)$. It must go through v_1 and v_3; and it must omit v_2 and v_4. Let $\gamma(v_1,v_3)$ be the section of $\gamma(a,b,a,b)$ between v_1 and v_3. We have two cases: v_1 appears before v_3 on $\gamma(a,b,a,b)$; or v_3 appears before v_1 on $\gamma(a,b,a,b)$. If the edge v_3 appears before v_1 on $\gamma(a,b,a,b)$ we denote by $\tilde{\gamma}^+(a,b,a,b)$ be the section of $\gamma(a,b,a,b)$ after v_1. The curve $\gamma^-(a,a,a,a)\cup\{v_1\}\cup\tilde{\gamma}^+(a,b,a,b)$ cannot be a geodesic on $\sigma^{(a,b,a,b)}(\omega)$ because it doesn't go through v_3. Hence,

$$T(\gamma^-(b,b,a,a)) + a + T(\gamma(v_1,v_3)) \quad < \quad T(\gamma^-(a,a,a,a)). \tag{49}$$

However, this contradicts the fact that $\gamma(a,a,a,a)$ is a geodesic on $\sigma^{(a,a,a,a)}(\omega)$. Hence, we conclude that v_1 is before v_3 on $\sigma^{(a,b,a,b)}(\omega)$.

In an analogous way we conclude that v_2 is after v_4 on $\sigma^{(b,a,b,a)}(\omega)$.

Let $\gamma^-(b,a,b,a)$ be the section of the geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$ before v_4. Define $\gamma^+(a,b,a,b)$ as the section of the geodesic on $\sigma^{(a,b,a,b)}(\omega)$ after v_3. We may assume that $\gamma^-(a,b,a,b) = \gamma^-(a,a,a,a)$ and $\gamma^+(b,a,b,a) = \gamma^+(a,a,a,a)$. Let us denote $\gamma(v_1,v_2)$ the section of $\gamma(a,a,a,a)$ between v_1 and v_2. Let e_{12} be the percolation time over $\gamma(v_1,v_2)$. Define $\gamma(v_3,v_4)$ the section of $\gamma(b,b,a,a,)$ between v_3 and v_4 and $e_{34} = T(\gamma(v_3,v_4))$. Let $e_{13} = T(\gamma(v_1,v_3))$, where $\gamma(v_1,v_3)$ is the section between v_1 and v_3 on $\gamma(a,b,a,b)$. Similarly, let $e_{24} = T(\gamma(v_2,v_4))$, where $\gamma(v_2,v_4)$ is the section of $\gamma(b,a,b,a)$ between v_4 and v_2. Define

$$\rho \quad = \quad L_1 + e_{12} + R_2 + 2a. \tag{50}$$

The shortest passage times on $\sigma^{(a,a,a,b)}(\omega)$, $\sigma^{(a,a,b,a)}(\omega)$, and $\sigma^{(a,a,b,b)}(\omega)$ are all equal to the shortest passage time on $\sigma^{(a,a,a,a)}(\omega)$, i.e.

$$\begin{aligned} f(\sigma^{(a,a,a,a)}(\omega)) &= f(\sigma^{(a,a,b,b)}(\omega)) = f(\sigma^{(a,a,a,b)}(\omega)) = f(\sigma^{(a,a,b,a)}(\omega)) \\ &= \rho. \end{aligned} \tag{51}$$

If we take the path $\gamma(a,a,a,a)$ on the environment $\sigma^{a,b,b,b}(\omega)$, the passage time would be $\rho+(b-a)$. Hence,

$$f(\sigma^{(a,b,b,b)}(\omega)) \le \rho+(b-a). \tag{52}$$

Monotonicity implies that the following relations are satisfied

$$f(\sigma^{(a,b,a,a)}(\omega)) \le f(\sigma^{(a,b,b,a)}(\omega)); \tag{53}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \le f(\sigma^{(b,a,a,b)}(\omega)); \tag{54}$$
$$f(\sigma^{(b,a,b,b)}(\omega)) \le f(\sigma^{(b,b,b,b)}(\omega)). \tag{55}$$

On the environment $\sigma^{(b,b,a,a)}(\omega)$, the passage time over $\gamma(b,b,a,a)$ is equal to the sum of the passage times over the sections $\gamma^-(b,b,a,a)$, $\gamma(v_3,v_4)$, $\gamma^+(b,b,a,a)$, and the passage times of edges v_3 and v_4. Therefore,

$$f(\sigma^{(b,b,a,a)}(\omega)) \quad = \quad T(\gamma^-(b,b,a,a))+T(\gamma^+(b,b,a,a))+e_{34}+2a. \tag{56}$$

The curve $\gamma(a,b,a,b)$ has the same passage time as the curve obtained by concatenating $\gamma^-(a,a,a,a)$, $\{v_1\}$, $\gamma(v_1,v_3)$, $\{v_3\}$, and $\gamma^+(a,b,a,b)$. Therefore,

$$f(\sigma^{(a,b,a,b)}(\omega)) \quad = \quad L_1+e_{13}+T(\gamma^+(a,b,a,b))+2a. \tag{57}$$

Using an analogous observation, we obtain

$$f(\sigma^{(b,a,b,a)}(\omega)) \quad = \quad R_2+e_{24}+T(\gamma^-(b,a,b,a))+2a. \tag{58}$$

We will obtain an upper bound for $f(\sigma^{(b,b,a,b)}(\omega))$ using the curve obtained by concatenating $\gamma^-(b,b,a,a)$, $\{v_3\}$, and $\gamma^+(a,b,a,b)$. Similarly, we will derive an upper bound for $f(\sigma^{(b,b,b,a)}(\omega))$ by calculating the passage time over the curve obtained by concatenating $\gamma^-(b,a,b,a)$, $\{v_4\}$, and $\gamma^+(b,b,a,a)$.

$$f(\sigma^{(b,b,a,b)}(\omega)) \quad \le \quad T(\gamma^-(b,b,a,a))+T(\gamma^+(a,b,a,b))+a; \tag{59}$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \quad \le \quad T(\gamma^-(b,a,b,a))+T(\gamma^+(b,b,a,a))+a. \tag{60}$$

The relations (51–60) can be used with (4) to obtain

$$\partial_S f(\omega) \quad \ge \quad -\rho-(b-a)+4a+L_1+R_2+e_{13}+e_{34}+e_{24}.$$

The last equation can be further simplified using the definition of ρ from (50). We derive

$$\partial_S f(\omega) \quad \ge \quad -(b-a)+2a+e_{13}+e_{34}+e_{24}-e_{12}. \tag{61}$$

It remains to observe that on $\sigma^{(a,a,a,a)}(\omega)$, the geodesic $\gamma^{(a,a,a,a)}$ must have the passage time that is shorter than or equal to the passage time over the concatenation of $\gamma^-(a,a,a,a)$, $\{v1\}$, $\gamma(v_1,v_3)$, $\{v_3\}$, $\gamma(v_3,v_4)$, $\{v_4\}$, $\gamma(v_2,v_4)$, $\{v_2\}$, $\gamma^+(a,a,a,a)$. Therefore,

$$e_{12} \quad \le \quad 2a+e_{13}+e_{34}+e_{24}.$$

The last inequality and (61) further imply that $\partial_S f(\omega) \ge -(b-a)$. As a consequence, we have $\partial_S f(\omega) \ge -2(b-a)$. □

Proposition 12. *Assume that there is a geodesic on $\sigma^{(a,a,a,a)}(\omega)$ that omits one element of S. Then $\partial_S f(\omega) \ge -2(b-a)$.*

Proof. Let $\gamma(a,a,a,a)$ be a geodesic that does not pass through v_4. We may assume that it passes through v_1, v_2, and v_3. Otherwise, the desired inequality would be a consequence of Propositions 10 and 11. The pair $((a,a,a,a),(a,a,a,b))$ is a negatively helpful pair. Consider the following 8 relations.

$$
\begin{aligned}
&\sigma^{(a,a,b,b)}(\omega) \in \quad E_1 \cap E_2 \qquad \cap \hat{E}_4^C; \quad && \sigma^{(a,b,a,b)}(\omega) \in \quad E_1 \qquad \cap E_3 \cap \hat{E}_4^C;\\
&\sigma^{(a,b,b,a)}(\omega) \in \quad E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4; \quad && \sigma^{(b,a,a,b)}(\omega) \in \qquad\quad E_2 \cap E_3 \cap \hat{E}_4^C;\\
&\sigma^{(b,a,b,a)}(\omega) \in \quad \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4; \quad && \sigma^{(b,b,a,a)}(\omega) \in \quad \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4.
\end{aligned} \tag{62}
$$

If any of them is not satisfied, then according to Proposition 3, there would be another negatively helpful pair that is disjoint with $((a,a,a,a),(a,a,a,b))$. The Proposition 8 would imply that $\partial_S f(\omega)$ is bounded below by $-2(b-a)$. Therefore, it remains to only consider the case in which all 8 of the relations (62) are satisfied.

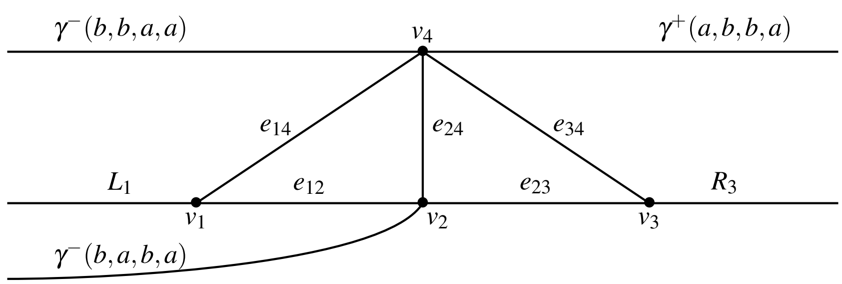
Assume that v_1, v_2, and v_3 appear in this order on the geodesic $\gamma(a,a,a,a)$. Let us define $\gamma^-(a,a,a,a)$, $\gamma(v_1,v_2)$, $\gamma(v_2,v_3)$, and $\gamma^+(a,a,a,a)$ as the sections in which v_1, v_2, and v_3 divide the curve $\gamma(a,a,a,a)$. These sections are defined to not contain the points v_1, v_2, and v_3. Let $\gamma(a,b,b,a)$ be a geodesic on $\sigma^{(a,b,b,a)}(\omega)$. We first prove that v_1 appears before v_4 on $\gamma(a,b,b,a)$. If this were not true, and if v_4 appeared before v_1, then the curve obtained by concatenating the section of $\gamma(a,b,b,a)$ before v_1 would have a strictly shorter passage time than $\gamma^-(a,a,a,a)$. The inequality would be strict because a geodesic on $\sigma^{(a,b,b,a)}(\omega)$ cannot omit v_4. However, this is a contradiction: there could not be a more efficient path to the edge v_1 than $\gamma^-(a,a,a,a)$ because $\gamma(a,a,a,a)$ is a geodesic on $\sigma^{(a,a,a,b)}(\omega)$. This proves that v_1 is before v_4 on $\sigma^{(a,b,b,a)}(\omega)$. In an analogous way, if we denote by $\gamma(b,b,a,a)$ a geodesic on $\sigma^{(b,b,a,a)}(\omega)$, we obtain that v_3 is after v_4 on $\gamma(b,b,a,a)$.

We may assume that the section of $\gamma(a,b,b,a)$ before v_1 coincides with $\gamma^-(a,a,a,a)$. In a similar way, we may assume that the section of $\gamma(b,b,a,a)$ after v_3 coincides with $\gamma^+(a,a,a,a)$. Let $\gamma^-(b,b,a,a)$ be the section of $\gamma(b,b,a,a)$ before v_4. Let $\gamma^+(a,b,b,a)$ be the section of $\gamma(a,b,b,a)$ after v_4.

Let $\gamma(b,a,b,a)$ be a geodesic on $\sigma^{(b,a,b,a)}(\omega)$. There are two cases: v_2 can be before or after v_4 on $\gamma(b,a,b,a)$. The two cases are analogous. We will assume, without loss of generality, that v_2 is before v_4. We may assume that the section of $\gamma(b,a,b,a)$ after v_4 coincides with $\gamma^+(a,b,b,a)$. Let $\gamma^-(b,a,b,a)$ be the section of $\gamma(b,a,b,a)$ before v_2.

Let $\gamma(v_1,v_4)$ be the section of $\gamma(a,b,b,a)$ between v_1 and v_4; $\gamma(v_3,v_4)$ the section of $\gamma(b,b,a,a)$ between v_3 and v_4; and $\gamma(v_2,v_4)$ the section of $\gamma(b,a,b,a)$ between v_2 and v_4. Define the numbers $L_1 = T(\gamma^-(a,a,a,a))$, $e_{12} = T(\gamma(v_1,v_2))$, $e_{23} = T(\gamma(v_2,v_3))$, $R_3 = T(\gamma^+(a,a,a,a))$, $e_{14} = T(\gamma(v_1,v_4))$, $e_{24} = T(\gamma(v_2,v_4))$, $e_{34} = T(\gamma(v_3,v_4))$, and

$$\rho \quad = \quad L_1 + e_{12} + e_{23} + R_3 + 3a. \tag{63}$$



The following identities follow directly from the definitions of the curves $\gamma(a,a,a,a)$, $\gamma(a,b,b,a)$, and $\gamma(b,a,b,a)$, and $\gamma(b,b,a,a)$.

$$f(\sigma^{(a,a,a,a)}(\omega)) = f(\sigma^{(a,a,a,b)}(\omega)) = \rho; \quad (64)$$

$$f(\sigma^{(b,a,b,a)}(\omega)) = T(\gamma^-(b,a,b,a)) + e_{24} + T(\gamma^+(a,b,b,a)) + 2a; \quad (65)$$

$$f(\sigma^{(b,b,a,a)}(\omega)) = R_3 + e_{34} + T(\gamma^-(b,b,a,a)) + 2a. \quad (66)$$

We can use monotonicity to obtain

$$f(\sigma^{(a,a,b,a)}(\omega)) \leq f(\sigma^{(a,b,b,a)}(\omega)); \quad f(\sigma^{(b,a,a,a)}(\omega)) \leq f(\sigma^{(b,a,a,b)}(\omega)), \quad (67)$$

$$f(\sigma^{(b,b,a,b)}(\omega)) \leq f(\sigma^{(a,b,a,b)}(\omega)) + (b-a); \quad f(\sigma^{(a,b,b,b)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \quad (68)$$

Let $\gamma(a,a,b,b)$ be a geodesic on $\sigma^{(a,a,b,b)}(\omega)$. From (62) we know that $\sigma^{(a,a,b,b)}(\omega) \in \hat{E}_1 \cap E_2 \cap \hat{E}_4^C$. Therefore, we are sure that $\gamma(a,a,b,b)$ passes through v_1 and v_3 and that it omits v_4. However, we don't know whether $\gamma(a,a,b,b)$ passes through v_3. Let us denote by $\gamma^+(a,a,b,b)$ the section of $\gamma(a,a,b,b)$ after v_2. We may assume that the section of $\gamma(a,a,b,b)$ before v_2 consides with $\gamma(a,a,a,a)$. The section $\gamma^+(a,a,b,b)$ may or may not pass through v_3. Therefore, the passage time over $\gamma^+(a,a,b,b)$ on the environments $\sigma^{\overrightarrow{r}}(\omega)$ may depend on $\overrightarrow{r} \in \{a,b\}^4$. We have the equatality

$$f(\sigma^{(a,a,b,b)}(\omega)) = L_1 + e_{12} + 2a + T(\gamma^+(a,a,b,b), \sigma^{(a,a,b,b)}(\omega)). \quad (69)$$

We will now find a bound for $f(\sigma^{(b,a,b,b)}(\omega))$. Consider the concatenation of $\gamma^-(b,a,b,a)$, $\{v_2\}$, and $\gamma^+(a,a,b,b)$. The passage time over this curve is also an upper bound for $f(\sigma^{(b,a,b,b)}(\omega))$. Hence,

$$\begin{aligned} f(\sigma^{(b,a,b,b)}(\omega)) &\leq T(\gamma^-(b,a,b,a)) + a + T(\gamma^+(a,a,b,b), \sigma^{(b,a,b,b)}(\omega)) \\ &= T(\gamma^-(b,a,b,a)) + a + T(\gamma^+(a,a,b,b), \sigma^{(a,a,b,b)}(\omega)). \quad (70) \end{aligned}$$

The last equality holds because the two environments $\sigma^{(a,a,b,b)}(\omega)$ and $\sigma^{(b,a,b,b)}(\omega)$ differ only in the value that is assigned to the edge v_1. However, the curve $\gamma^+(a,a,b,b)$ does not pass through v_1.

The passage time over the curve $\gamma^-(b,b,a,a) \cup \{v_4\} \cup \gamma^+(a,b,b,a)$ is an upper bound for $f(\sigma^{(b,b,b,a)}(\omega))$. The passage time over the concatenation of the curves $\gamma^-(a,a,a,a)$, $\{v_1\}$, $\gamma(v_1,v_4)$, $\{v_4\}$, $\gamma(v_3,v_4)$, $\{v_3\}$, $\gamma^+(a,a,a,a)$ is an upper bound for $f(\sigma^{(a,b,a,a)}(\omega))$. Therefore,

$$f(\sigma^{(b,b,b,a)}(\omega)) \leq T(\gamma^-(b,b,a,a)) + T(\gamma^+(a,b,b,a)) + a; \quad (71)$$

$$f(\sigma^{(a,b,a,a)}(\omega)) \leq L_1 + e_{14} + e_{34} + R_3 + 3a. \quad (72)$$

We now use the formula (4) and the relations (64–72) to obtain

$$\partial_S f(\omega) \geq -(b-a) + a + e_{24} + e_{12} - e_{14}. \quad (73)$$

Let us consider the environment $\sigma^{(a,b,b,a)}(\omega)$ once again and derive a triangle inequality that will finish the proof of this proposition. The geodesic $\gamma(a,b,b,a)$ must have a passage time that is smaller than or equal than the passage time over the curve in which the segment $\gamma(v_1,v_4)$ of $\gamma(a,b,b,a)$ is replaced by $\gamma(v_1,v_2) \cup \{v_2\} \cup \gamma(v_2,v_4)$. Since the value assigned to v_2 is b on the environment $\sigma^{(a,b,b,a)}(\omega)$, we have

$$e_{14} \leq e_{12} + e_{24} + b.$$

The last inequality is equivalent to $e_{12} + e_{24} - e_{14} \geq -b$, which together with (73) implies that $\partial_S f(\omega) \geq -2(b-a)$. □

4.2. Case in which there are no negatively helpful pairs.

Proposition 13. *If there are no negatively helpful pairs, then all of the following relations must be satisfied*

$$\sigma^{(a,a,b,b)}(\omega) \in E_1 \cap E_2 \cap \hat{E}_3^C \cap \hat{E}_4^C; \quad \sigma^{(a,b,a,b)}(\omega) \in E_1 \cap \hat{E}_2^C \cap E_3 \cap \hat{E}_4^C;$$
$$\sigma^{(a,b,b,a)}(\omega) \in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4; \quad \sigma^{(b,a,a,b)}(\omega) \in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C;$$
$$\sigma^{(b,a,b,a)}(\omega) \in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4; \quad \sigma^{(b,b,a,a)}(\omega) \in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4;$$
$$\sigma^{(a,a,a,a)}(\omega) \in E_1 \cap E_2 \cap E_3 \cap E_4.$$

Proof. This is an immediate consequence of Proposition 3. □

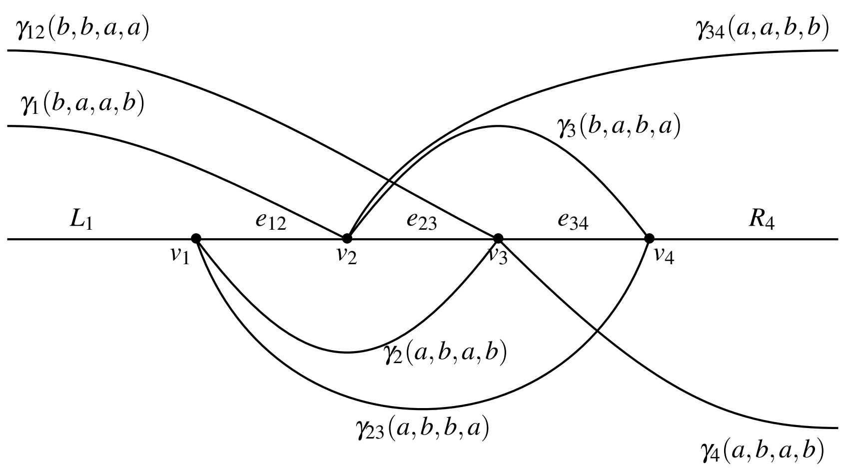
Proposition 14. *If there are no negatively helpful pairs, then $\partial_S f(\omega) \geq -(b-a)$.*

Proof. From Proposition 13 we know that every geodesic on $\sigma^{(a,a,a,a)}(\omega)$ passes through all of elements of S. Assume that $\gamma(a,a,a,a)$ is one geodesic on $\sigma^{(a,a,a,a)}(\omega)$. Without loss of generality, we may assume that the order of the edges of S is v_1, v_2, v_3, v_4. Let L_1 be the first passage percolation time over the section of $\gamma(a,a,a,a)$ before the edge v_1. Let e_{12} be the first passage percolation time between v_1 and v_2; e_{23} the time between v_2 and v_3; e_{34} the time between v_3 and v_4; and R_4 the time after v_4. Let us define

$$\rho \quad = \quad L_1 + e_{12} + e_{23} + e_{34} + R_4 + 4a. \tag{74}$$

Then

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad T(\gamma(a,a,a,a),\omega) = \rho. \tag{75}$$



We know that the condition $\sigma^{(a,a,b,b)}(\omega) \in E_1 \cap E_2 \cap \hat{E}_3^C \cap \hat{E}_4^C$ must be satisfied. Let $\hat{\gamma}(a,a,b,b)$ be a geodesic on $\sigma^{(a,a,b,b)}(\omega)$. This geodesic must pass through v_1 and v_2 and must not pass through either of v_3 and v_4. It is easy to prove that v_1 and v_2 must occur in the same order on $\hat{\gamma}(a,a,b,b)$ as they do on $\gamma(a,a,a,a)$. The paths $\hat{\gamma}(a,a,b,b)$ and $\gamma(a,a,a,a)$ must have the same costs before v_2. Therefore, we may assume that they coalesce before v_2. Let $\gamma_{34}(a,a,b,b)$ be the section of $\hat{\gamma}(a,a,b,b)$ after the edge v_2. The section $\gamma_{34}(a,a,b,b)$ contains no edges from S. We will thus omit ω and just write $T(\gamma_{34}(a,a,b,b))$. Since $\hat{\gamma}(a,a,b,b)$ must not be a geodesic on $\sigma^{(a,a,a,a)}(\omega)$, and $\gamma(a,a,a,a)$ must not be a geodesic on $\sigma^{(a,a,b,b)}(\omega)$, we conclude that there exists a real number $\hat{\theta}_{34}(a,a,b,b)$ from the interval $(0,b-a)$ such that

$$T(\gamma_{34}(a,a,b,b)) = e_{23}+e_{34}+R_4+2a+2\theta_{34}(a,a,b,b).$$

Consequently,

$$f(\sigma^{(a,a,b,b)}(\omega)) = \rho+2\theta_{34}(a,a,b,b). \tag{76}$$

We will now use that $\sigma^{(a,b,a,b)}(\omega) \in E_1 \cap \hat{E}_2^C \cap E_3 \cap \hat{E}_4^C$. Let $\hat{\gamma}(a,b,a,b)$ be a geodesic on $\sigma^{(a,b,a,b)}(\omega)$. Using a similar reasoning as before, we conclude that we may assume that $\hat{\gamma}(a,b,a,b)$ coalesce with $\gamma(a,a,a,a)$ before v_1 and that there are two sections $\gamma_2(a,b,a,b)$ and $\gamma_4(a,b,a,b)$ of $\hat{\gamma}(a,b,a,b)$ such that $\gamma_2(a,b,a,b)$ connects v_1 and v_3 and $\gamma_4(a,b,a,b)$ starts at v_3 and contains the remainder of the curve $\hat{\gamma}(a,b,a,b)$. Using a similar argument to the one before, we conclude that there is $\theta_2(a,b,a,b) \in (0,b-a)$ and $\theta_4(a,b,a,b) \in (0,b-a)$ such that

$$\begin{aligned} T(\gamma_2(a,b,a,b)) &= e_{12}+e_{23}+a+\theta_2(a,b,a,b); \\ T(\gamma_4(a,b,a,b)) &= R_4+e_{34}+a+\theta_4(a,b,a,b); \\ f(\sigma^{(a,b,a,b)}) &= \rho+\theta_2(a,b,a,b)+\theta_4(a,b,a,b). \end{aligned} \tag{77}$$

By analyzing the requirement $\sigma^{(a,b,b,a)}(\omega) \in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4$ and applying similar logic, we conclude that there exists a path $\gamma_{23}(a,b,b,a)$ that connects v_1 and v_4 and that there exists $\theta_{23}(a,b,b,a) \in (0,b-a)$ such that the following equations are satisfied

$$\begin{aligned} T(\gamma_{23}(a,b,b,a)) &= e_{12}+e_{23}+e_{34}+2a+2\theta_{23}(a,b,b,a); \\ f(\sigma^{(a,b,b,a)}(\omega)) &= \rho+2\theta_{23}(a,b,b,a). \end{aligned} \tag{78}$$

Let us now analyze the requirement $\sigma^{(b,a,a,b)} \in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C$. Let $\hat{\gamma}(b,a,a,b)$ be a geodesic on $\sigma^{(b,a,a,b)}(\omega)$. This geodesic must pass through v_3 and omit v_4. This property was also satisfied by the geodesic $\hat{\gamma}(a,b,a,b)$. Therefore, these two geodesics must have the same passage time after v_3 and we may assume that they coalesce. Hence, we may assume that after v_3, the geodesic $\hat{\gamma}(b,a,a,b)$ goes along the path $\gamma_4(a,b,a,b)$. Let us denote by $\gamma_1(b,a,a,b)$ the section of $\hat{\gamma}(b,a,a,b)$ before the edge v_2. The section $\gamma_1(b,a,a,b)$ must not pass through v_1. Using the same reasoning as before, we conclude that there exists $\theta_1(b,a,a,b) \in (0,b-a)$ such that

$$\begin{aligned} T(\gamma_1(b,a,a,b)) &= L_1+e_{12}+a+\theta_1(b,a,a,b); \\ f(\sigma^{(b,a,a,b)}(\omega)) &= \rho+\theta_1(b,a,a,b)+\theta_4(a,b,a,b). \end{aligned} \tag{79}$$

Using a similar reasoning, we conclude that there is a geodesic $\hat{\gamma}(b,a,b,a)$ on $\sigma^{(b,a,b,a)}(\omega)$ that coincides with $\gamma_1(b,a,a,b)$ before the edge v_2 and that follows the path $\gamma_3(b,a,b,a)$ that connects v_2 and v_4 but avoids v_3. We also conclude that there is $\theta_3(b,a,b,a) \in (0,b-a)$ for which

$$\begin{aligned} T(\gamma_3(b,a,b,a)) &= e_{23}+e_{34}+a+\theta_3(b,a,b,a); \\ f(\sigma^{(b,a,b,a)}(\omega)) &= \rho+\theta_1(b,a,a,b)+\theta_3(b,a,b,a). \end{aligned} \tag{80}$$

By analyzing $\sigma^{(b,b,a,a)}(\omega) \in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4$ and the geodesic $\hat{\gamma}(b,b,a,a)$ on the environment $\sigma^{(b,b,a,a)}(\omega)$, we conclude that there is a path $\gamma_{12}(b,b,a,a)$ that connects the source with v_3 and omits v_1 and v_2. There is a real number $\theta_{12}(b,b,a,a) \in (0,b-a)$ such that

$$\begin{aligned} T(\gamma_{12}(b,b,a,a)) &= L_1+e_{12}+e_{23}+2a+2\theta_{12}(b,b,a,a); \\ f(\sigma^{(b,b,a,a)}(\omega)) &= \rho+2\theta_{12}(b,b,a,a). \end{aligned} \tag{81}$$

We will now obtain an upper bound for $f(\sigma^{(a,a,a,b)}(\omega))$. Consider the section of the curve $\gamma(a,a,a,a)$ before the edge v_3. When this section is concatenated with v_3 and $\gamma_4(a,b,a,b)$ we obtain the curve that we will call $\gamma(a,a,a,b)$. The passage time over $\gamma(a,a,a,b)$ is at $\rho+\theta_4(a,b,a,b)$, hence

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho+\theta_4(a,b,a,b). \tag{82}$$

Using similar arguments we obtain the following inequalities

$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho+\theta_3(b,a,b,a); \tag{83}$$

$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho+\theta_2(a,b,a,b); \tag{84}$$

$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho+\theta_1(b,a,a,b). \tag{85}$$

The monotonicity implies the following bound

$$f(\sigma^{(a,b,b,b)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \tag{86}$$

Let us consider the environment $\sigma^{(b,a,b,b)}(\omega)$. Define the curve $\gamma(b,a,b,b)$ as the concatenation of $\gamma_1(b,a,a,b)$, $\{v_2\}$, and $\gamma_{34}(a,a,b,b)$. The passage time over the curve $\gamma(b,a,b,b)$ is $\rho+\theta_1(b,a,a,b)+2\theta_{34}(a,a,b,b)$. Similarly, on the environment $\sigma^{(b,b,a,b)}(\omega)$, the passage time over the concatenation of $\gamma_{12}(b,b,a,a)$, $\{v_3\}$, and $\gamma_4(a,b,a,b)$ is equal to $\rho+2\theta_{12}(b,b,a,a)+\theta_4(a,b,a,b)$. On the environment $\sigma^{(b,b,b,a)}(\omega)$ we consider the following curve $\gamma(b,b,b,a)$. First we take the section of $\gamma(a,a,a,a)$ before v_1, then we include the edge v_1 (which has unfavorable passage time b), then we include the path $\gamma_{23}(a,b,b,a)$, then the edge v_4, and the remainder of the curve $\gamma(a,a,a,a)$. The passage time over the constructed curve $\gamma(b,b,b,a)$ is $T(\gamma(b,b,b,a),\sigma^{(b,b,b,a)}(\omega))=\rho+(b-a)+2\theta_{23}(a,b,b,a)$. We have established the following inequalities

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho+\theta_1(b,a,a,b)+2\theta_{34}(a,a,b,b); \tag{87}$$

$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho+2\theta_{12}(b,b,a,a)+\theta_4(a,b,a,b); \tag{88}$$

$$f(\sigma^{(b,b,b,a)}(\omega)) \leq \rho+(b-a)+2\theta_{23}(a,b,b,a). \tag{89}$$

We now use the formulas (75–89) to obtain $\partial_S f(\omega)\geq -(b-a)$. □

5. COMPUTER-ASSISTED PROOF OF THE INEQUALITY $\mathscr{L}_4 \geq -2$

5.1. **Assumptions that we are allowed to make.** Throughout this section we will assume that the environment ω and the four-element set $S\subseteq W$ are fixed. The inequality $\partial_S f(\omega)\geq -2(b-a)$ has been already established in multiple cases. Let us summarize these cases:

$$\sigma^{(a,a,a,a)}(\omega)\notin E_1\cap E_2\cap E_3\cap E_4 \implies \partial_S f(\omega)\geq -2(b-a); \tag{90}$$

$$\text{There are no negatively helpful pairs} \implies \partial_S f(\omega)\geq -2(b-a). \tag{91}$$

Therefore, it remains to consider the cases in which we are allowed to assume that the environment $\sigma^{(a,a,a,a)}(\omega)$ belongs to each of the events E_1, E_2, E_3, and E_4. We will choose one geodesic on $\gamma^{(a,a,a,a)}(\omega)$ and denote it by $\gamma(a,a,a,a)$. This geodesic passes through all elements of S, because all geodesics must do so on $\sigma^{a,a,a,a}(\omega)$. Without loss of generality, we may assume that the edges v_1, v_2, v_3, and v_4 appear in this order. These edges divide the curve $\gamma(a,a,a,a)$ into sections $\gamma^-(a,a,a,a)$, $\gamma(v_1,v_2)$, $\gamma(v_2,v_3)$, $\gamma(v_3,v_4)$, and

$\gamma^+(a,a,a,a)$. We define these sections in such a way that they do not contain their endpoints v_1, v_2, v_3, v_4. Then the passage times over these sections are the same on all environments $\sigma^{\overrightarrow{r}}(\omega)$ for $\overrightarrow{r} \in \{a,b\}^4$. We will denote these passage times by L_1, e_{12}, e_{23}, e_{34}, and R_4, respectively. Define

$$\rho \quad = \quad 4a + L_1 + e_{12} + e_{23} + e_{34} + R_4. \tag{92}$$

The value of $f(\sigma^{(a,a,a,a)}(\omega))$ is ρ.

5.2. **Helpfulness indicators.** For each $i \in \{1,2,3,4\}$, we define the helpfulness indicator $\chi_i : \{a,b\}^4 \to \{0,1\}$ in the following way

$$\chi_i(\overrightarrow{r}) \quad = \quad \begin{cases} 0, & \text{if } r_i = a \text{ and } \sigma^{\overrightarrow{r}}(\omega) \in E_i, \\ 0, & \text{if } r_i = b \text{ and } \sigma^{\overrightarrow{r}}(\omega) \in \hat{E}_i^C, \\ 1, & \text{if } r_i = a \text{ and } \sigma^{\overrightarrow{r}}(\omega) \in E_i^C, \\ 1, & \text{if } r_i = b \text{ and } \sigma^{\overrightarrow{r}}(\omega) \in \hat{E}_i. \end{cases} \tag{93}$$

If the helpfulness indicator is equal to 1, then we can easily identify one very convenient negatively helpful pair, due to Proposition 3.

If all helpfulness indicators of $\overrightarrow{r}$ are 0, then we can gain a very valuable knowledge about geodesic on $\sigma^{\overrightarrow{r}}(\omega)$. Moreover, such knowledge would be sufficiently easy to gain in the case that $\overrightarrow{r}$ has exactly two as and two bs. We will build a computer algorithm capable of constructing geodesics on $\sigma^{\overrightarrow{r}}(\omega)$ when all of the helpfulness indicators of $\overrightarrow{r}$ are equal to 0.

6. Computer-generated proof

This section completes the proof of the inequality $\mathscr{L}_4 \geq -2$. The section is written by the computer, and although it is somewhat long, the reader can read and verify that it is correct. There are 24 possible cases. However, due to symmetry, we can reduce the casework to twelve cases. The other twelve are analogous. For example, the case $\sigma^{(a,a,b,b)}(\omega) \in E_1^C$ is analogous to $\sigma^{(b,b,a,a)}(\omega) \in E_4^C$; the case $\sigma^{(a,a,b,b)}(\omega) \in \hat{E}_3$ is analogous to the case $\sigma^{(b,b,a,a)}(\omega) \in \hat{E}_2$.

These 12 cases were checked by the computer program. In each of the cases the computer has identified the proof of $\partial_S f(\omega) \geq -2(b-a)$. The computer program generated the following human-readable proofs for each of the cases.

6.1. **Analysis of the case $\sigma^{(a,a,b,b)}(\omega) \in \hat{E}_4$.** The Proposition 3 implies

$$f(\sigma^{(a,a,b,b)}(\omega)) = (b-a) + f(\sigma^{(a,a,b,a)}(\omega)). \tag{94}$$

Initially, we can make the following straightforward conclusions about the environment.

$$\begin{aligned} \sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4; \\ \sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4; \\ \sigma^{(b,a,b,a)}(\omega) &\in E_2 \cap \hat{E}_3^C \cap E_4; \\ \sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_3^C \cap E_4; \\ \sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C; \\ \sigma^{(a,b,a,b)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap E_3 \cap \hat{E}_4^C. \end{aligned}$$

We will now prove that the previous equations imply one additional inclusion:

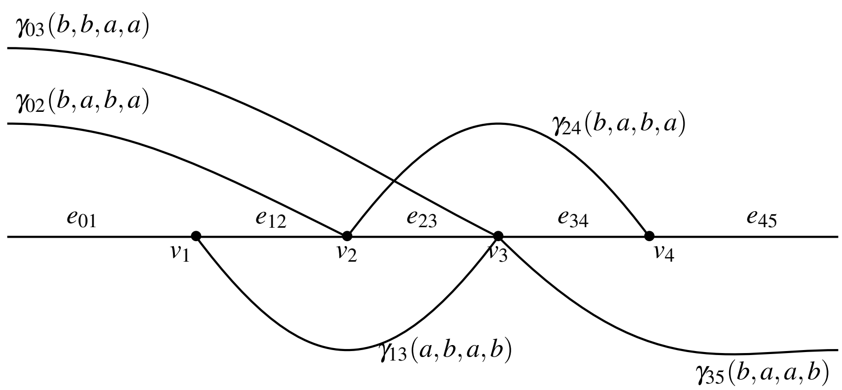
$$\sigma^{(b,a,b,a)}(\omega) \in \hat{E}_1^C.$$

We know that $\sigma^{(b,a,a,b)}(\omega)$ belongs to $\hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C$ and that $\sigma^{(b,a,b,a)}(\omega)$ belongs to $E_2 \cap \hat{E}_3^C \cap E_4$. Let us consider an arbitrary geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. We know that it must pass through v_2. The section before v_2 has the same passage time as the corresponding section of each geodesic on $\sigma^{(b,a,a,b)}(\omega)$. Hence, each geodesic on $\sigma^{(b,a,b,a)}(\omega)$ must omit v_1. Hence, we have $\sigma^{(b,a,b,a)}(\omega) \in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4$. The following relations are satisfied.

$$\begin{aligned} \sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4; \\ \sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4; \\ \sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4; \\ \sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_3^C \cap E_4; \\ \sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C; \\ \sigma^{(a,b,a,b)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap E_3 \cap \hat{E}_4^C. \end{aligned} \tag{95}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) = \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \tag{96}$$



Let us denote by $\gamma(a,b,a,b)$ a geodesic on the environment $\sigma^{(a,b,a,b)}(\omega)$. Let $\gamma_{13}(a,b,a,b)$ be the section of this geodesic between the edges v_1 and v_3. There exists a real number $\theta_{13}(a,b,a,b) \in (0, b-a)$ such that the following equality holds

$$T(\gamma_{13}(a,b,a,b), \tilde{\omega}) = e_{12} + e_{23} + a + \theta_{13}(a,b,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0, b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b), \tilde{\omega}) = e_{34} + e_{45} + a + \theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0, b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) = e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) = e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us summarize the conclusions that we can make by analyzing (95). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{13}(a,b,a,b),$$
$$\theta_{24}(b,a,b,a), \text{ and } \theta_{35}(b,a,a,b)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,b,a,b)}(\omega)) = \rho+\theta_{13}(a,b,a,b)+\theta_{35}(b,a,a,b), \tag{97}$$
$$f(\sigma^{(a,b,b,a)}(\omega)) \geq a-b+f(\sigma^{(a,b,b,b)}(\omega)), \tag{98}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) \geq a-b+f(\sigma^{(b,a,b,b)}(\omega)), \tag{99}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \tag{100}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho+2\theta_{03}(b,b,a,a). \tag{101}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho+\theta_{35}(b,a,a,b), \tag{102}$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq a-b+f(\sigma^{(a,a,b,b)}(\omega)), \tag{103}$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho+\theta_{13}(a,b,a,b), \tag{104}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a). \tag{105}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(b,a,a,b), \tag{106}$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \tag{107}$$

Combining the equations and inequalities (96–107), we derive

$$\partial_S f(\omega) \geq a-b+\theta_{24}(b,a,b,a)-\theta_{35}(b,a,a,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,a,b,b)}(\omega) \in \hat{E}_4$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.2. **Analysis of the case $\sigma^{(a,a,b,b)}(\omega)\in\hat{E}_3$.** The Proposition 3 implies

$$f(\sigma^{(a,a,b,b)}(\omega))=(b-a)+f(\sigma^{(a,a,a,b)}(\omega)). \tag{108}$$

Initially, we can make the following straightforward conclusions about the environment.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap\hat{E}_2^C\cap E_3\cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap E_2\cap\hat{E}_3^C\cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1\cap\hat{E}_2^C\cap\hat{E}_3^C\cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in E_2\cap E_3\cap\hat{E}_4^C;\\
\sigma^{(a,b,a,b)}(\omega) &\in E_1\cap E_3\cap\hat{E}_4^C.
\end{aligned}$$

We will now prove that the previous equations imply one additional inclusion:

$$\sigma^{(b,a,a,b)}(\omega)\in\hat{E}_1^C.$$

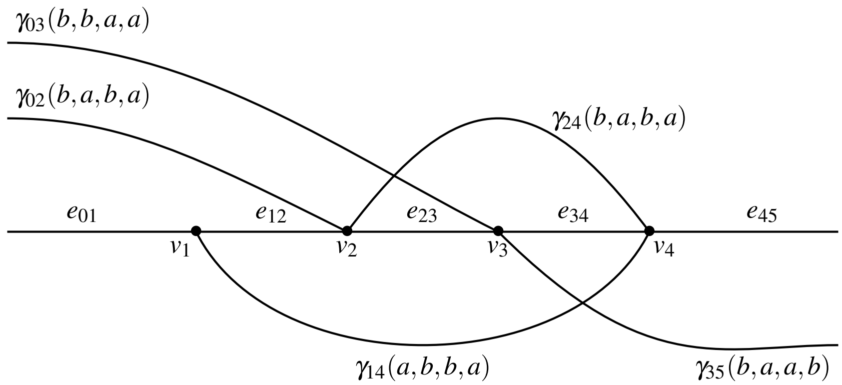
We know that $\sigma^{(b,a,b,a)}(\omega)$ belongs to $\hat{E}_1^C\cap E_2\cap E_3\cap\hat{E}_4^C$ and that $\sigma^{(b,a,a,b)}(\omega)$ belongs to $E_2\cap E_3\cap\hat{E}_4^C$. Let us consider an arbitrary geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. We know that it must pass through v_2. The section before v_2 has the same passage time as the corresponding section of each geodesic on $\sigma^{(b,a,b,a)}(\omega)$. Hence, each geodesic on $\sigma^{(b,a,a,b)}(\omega)$ must omit v_1.

Hence, we have $\sigma^{(b,a,a,b)}(\omega)\in\hat{E}_1^C\cap E_2\cap E_3\cap\hat{E}_4^C$. The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap\hat{E}_2^C\cap E_3\cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap E_2\cap\hat{E}_3^C\cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1\cap\hat{E}_2^C\cap\hat{E}_3^C\cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C\cap E_2\cap E_3\cap\hat{E}_4^C;\\
\sigma^{(a,b,a,b)}(\omega) &\in E_1\cap E_3\cap\hat{E}_4^C.
\end{aligned} \tag{109}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i\in\{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i\in\{1,2,3\}$ and $j=i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho,\ \text{where } \rho=4a+e_{01}+e_{12}+e_{23}+e_{34}+e_{45}. \tag{110}$$



Let us denote by $\gamma(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\gamma_{14}(a,b,b,a)$ be the section of this geodesic between the edges v_1 and v_4. There exists a real number $\theta_{14}(a,b,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{14}(a,b,b,a),\tilde{\omega}) \quad = \quad e_{12}+e_{23}+e_{34}+2a+2\theta_{14}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{34}+e_{45}+a+\theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us summarize the conclusions that we can make by analyzing (109). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a),\theta_{03}(b,b,a,a),\theta_{14}(a,b,b,a),$$
$$\theta_{24}(b,a,b,a),\text{ and }\theta_{35}(b,a,a,b)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$\begin{aligned} f(\sigma^{(a,b,a,b)}(\omega)) &\geq a-b+f(\sigma^{(a,b,b,b)}(\omega)), &(111)\\ f(\sigma^{(a,b,b,a)}(\omega)) &= \rho+2\theta_{14}(a,b,b,a), &(112)\\ f(\sigma^{(b,a,a,b)}(\omega)) &\geq a-b+f(\sigma^{(b,a,b,b)}(\omega)), &(113)\\ f(\sigma^{(b,a,b,a)}(\omega)) &= \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), &(114)\\ f(\sigma^{(b,b,a,a)}(\omega)) &= \rho+2\theta_{03}(b,b,a,a). &(115)\end{aligned}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$\begin{aligned} f(\sigma^{(a,a,a,b)}(\omega)) &\leq a-b+f(\sigma^{(a,a,b,b)}(\omega)), &(116)\\ f(\sigma^{(a,a,b,a)}(\omega)) &\leq \rho+\theta_{24}(b,a,b,a), &(117)\\ f(\sigma^{(a,b,a,a)}(\omega)) &\leq \rho+2\theta_{14}(a,b,b,a), &(118)\\ f(\sigma^{(b,a,a,a)}(\omega)) &\leq \rho+\theta_{02}(b,a,b,a). &(119)\end{aligned}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$\begin{aligned} f(\sigma^{(b,b,a,b)}(\omega)) &\leq \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(b,a,a,b), &(120)\\ f(\sigma^{(b,b,b,a)}(\omega)) &\leq f(\sigma^{(b,b,b,b)}(\omega)). &(121)\end{aligned}$$

Combining the equations and inequalities (110–121), we derive

$$\partial_S f(\omega) \geq a-b-\theta_{35}(b,a,a,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,a,b,b)}(\omega)\in\hat{E}_3$, we have obtained that $\partial_S f(\omega)\geq -2(b-a)$.

6.3. **Analysis of the case $\sigma^{(a,a,b,b)}(\omega)\in E_2^C$.** The Proposition 3 implies

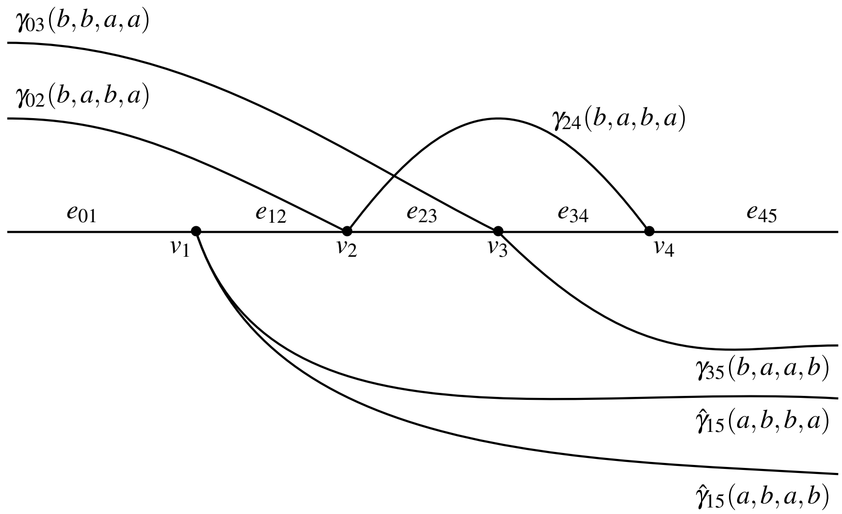
$$f(\sigma^{(a,a,b,b)}(\omega)) = f(\sigma^{(a,b,b,b)}(\omega)). \tag{122}$$

The following relations are satisfied.

$$\begin{aligned} \sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\ \sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap\hat{E}_2^C\cap E_3\cap E_4;\\ \sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap E_2\cap\hat{E}_3^C\cap E_4;\\ \sigma^{(a,b,b,a)}(\omega) &\in E_1\cap\hat{E}_2^C\cap\hat{E}_3^C;\\ \sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C\cap E_2\cap E_3\cap\hat{E}_4^C;\\ \sigma^{(a,b,a,b)}(\omega) &\in E_1\cap\hat{E}_2^C\cap\hat{E}_4^C. &(123)\end{aligned}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i\in\{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i\in\{1,2,3\}$ and $j=i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) = \rho, \text{ where } \rho = 4a+e_{01}+e_{12}+e_{23}+e_{34}+e_{45}. \tag{124}$$



Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) = e_{34}+e_{45}+a+\theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) = e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) = e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(a,b,a,b)$ a geodesic on the environment $\sigma^{(a,b,a,b)}(\omega)$. Let $\hat{\gamma}_{15}(a,b,a,b)$ be the section of this geodesic after the edge v_1. There exists a real number $\hat{\theta}_{15}(a,b,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{15}(a,b,a,b),\tilde{\omega}) = e_{12}+e_{23}+e_{34}+e_{45}+3a+2\hat{\theta}_{15}(a,b,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_3 = a$. Let us denote by $\hat{\gamma}(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\hat{\gamma}_{15}(a,b,b,a)$

be the section of this geodesic after the edge v_1. There exists a real number $\hat{\theta}_{15}(a,b,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{15}(a,b,b,a),\tilde{\omega}) \quad = \quad e_{12}+e_{23}+e_{34}+e_{45}+3a+2\hat{\theta}_{15}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_4 = a$. Let us summarize the conclusions that we can make by analyzing (123). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{24}(b,a,b,a),$$
$$\theta_{35}(b,a,a,b), \hat{\theta}_{15}(a,b,a,b), \text{ and } \hat{\theta}_{15}(a,b,b,a)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,b,a,b)}(\omega)) \quad = \quad \rho+2\hat{\theta}_{15}(a,b,a,b), \tag{125}$$
$$f(\sigma^{(a,b,b,a)}(\omega)) \quad = \quad \rho+2\hat{\theta}_{15}(a,b,b,a), \tag{126}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) \quad \geq \quad a-b+f(\sigma^{(b,a,b,b)}(\omega)), \tag{127}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) \quad = \quad \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \tag{128}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) \quad = \quad \rho+2\theta_{03}(b,b,a,a). \tag{129}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \quad \leq \quad \rho+2\hat{\theta}_{15}(a,b,a,b), \tag{130}$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \quad \leq \quad \rho+\theta_{24}(b,a,b,a), \tag{131}$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \quad \leq \quad \rho+2\hat{\theta}_{15}(a,b,b,a), \tag{132}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \quad \leq \quad \rho+\theta_{02}(b,a,b,a). \tag{133}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(a,b,b,b)}(\omega)) \quad \leq \quad f(\sigma^{(a,a,b,b)}(\omega)), \tag{134}$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \quad \leq \quad \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(b,a,a,b), \tag{135}$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \quad \leq \quad f(\sigma^{(b,b,b,b)}(\omega)). \tag{136}$$

Combining the equations and inequalities (124–136), we derive

$$\partial_S f(\omega) \quad \geq \quad a-b-\theta_{35}(b,a,a,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,a,b,b)}(\omega) \in E_2^C$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.4. **Analysis of the case $\sigma^{(a,a,b,b)}(\omega) \in E_1^C$.** The Proposition 3 implies

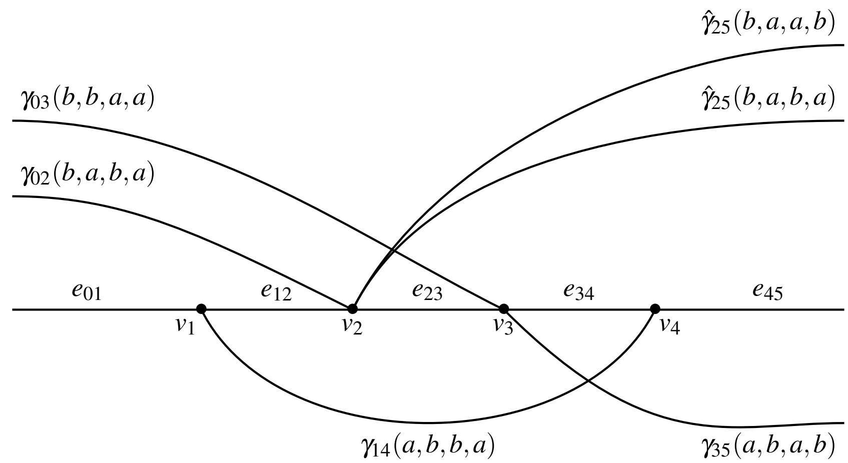
$$f(\sigma^{(a,a,b,b)}(\omega)) = f(\sigma^{(b,a,b,b)}(\omega)). \tag{137}$$

The following relations are satisfied.

$$\begin{aligned} \sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\ \sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap \hat{E}_2^C\cap E_3\cap E_4;\\ \sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap E_2\cap \hat{E}_3^C;\\ \sigma^{(a,b,b,a)}(\omega) &\in E_1\cap \hat{E}_2^C\cap \hat{E}_3^C\cap E_4;\\ \sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C\cap E_2\cap \hat{E}_4^C;\\ \sigma^{(a,b,a,b)}(\omega) &\in E_1\cap \hat{E}_2^C\cap E_3\cap \hat{E}_4^C. \end{aligned} \tag{138}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i\in\{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i\in\{1,2,3\}$ and $j=i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad=\quad \rho, \text{ where } \rho = 4a+e_{01}+e_{12}+e_{23}+e_{34}+e_{45}. \tag{139}$$



Let us denote by $\gamma(a,b,a,b)$ a geodesic on the environment $\sigma^{(a,b,a,b)}(\omega)$. Let $\gamma_{35}(a,b,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(a,b,a,b)\in(0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(a,b,a,b),\tilde{\omega}) \quad=\quad e_{34}+e_{45}+a+\theta_{35}(a,b,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\gamma_{14}(a,b,b,a)$ be the section of this geodesic between the edges v_1 and v_4. There exists a real number $\theta_{14}(a,b,b,a)\in(0,b-a)$ such that the following equality holds

$$T(\gamma_{14}(a,b,b,a),\tilde{\omega}) \quad=\quad e_{12}+e_{23}+e_{34}+2a+2\theta_{14}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a)\in(0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad=\quad e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\hat{\gamma}_{25}(b,a,a,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\hat{\theta}_{25}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{25}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+e_{45}+2a+\hat{\theta}_{25}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_3 = a$. Let us denote by $\hat{\gamma}(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\hat{\gamma}_{25}(b,a,b,a)$ be the section of this geodesic after the edge v_2. There exists a real number $\hat{\theta}_{25}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{25}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+e_{45}+2a+\hat{\theta}_{25}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_4 = a$. Let us summarize the conclusions that we can make by analyzing (138). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{14}(a,b,b,a),$$
$$\theta_{35}(a,b,a,b), \hat{\theta}_{25}(b,a,a,b), \text{ and } \hat{\theta}_{25}(b,a,b,a)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,b,a,b)}(\omega)) \quad \geq \quad a-b+f(\sigma^{(a,b,b,b)}(\omega)), \tag{140}$$
$$f(\sigma^{(a,b,b,a)}(\omega)) \quad = \quad \rho+2\theta_{14}(a,b,b,a), \tag{141}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) \quad = \quad \rho+\theta_{02}(b,a,b,a)+\hat{\theta}_{25}(b,a,a,b), \tag{142}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) \quad = \quad \rho+\theta_{02}(b,a,b,a)+\hat{\theta}_{25}(b,a,b,a), \tag{143}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) \quad = \quad \rho+2\theta_{03}(b,b,a,a). \tag{144}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \quad \leq \quad \rho+\hat{\theta}_{25}(b,a,a,b), \tag{145}$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \quad \leq \quad \rho+\hat{\theta}_{25}(b,a,b,a), \tag{146}$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \quad \leq \quad \rho+2\theta_{14}(a,b,b,a), \tag{147}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \quad \leq \quad \rho+\theta_{02}(b,a,b,a). \tag{148}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \quad \leq \quad f(\sigma^{(a,a,b,b)}(\omega)), \tag{149}$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \quad \leq \quad \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(a,b,a,b), \tag{150}$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \quad \leq \quad f(\sigma^{(b,b,b,b)}(\omega)). \tag{151}$$

Combining the equations and inequalities (139–151), we derive

$$\partial_S f(\omega) \quad \geq \quad a-b+\theta_{02}(b,a,b,a)-\theta_{35}(a,b,a,b).$$

Recall that each of the constants θ belongs to the interval $[0, b-a]$. Thus, under the condition $\sigma^{(a,a,b,b)}(\omega) \in E_1^C$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.5. **Analysis of the case $\sigma^{(a,b,a,b)}(\omega) \in \hat{E}_4$.** The Proposition 3 implies

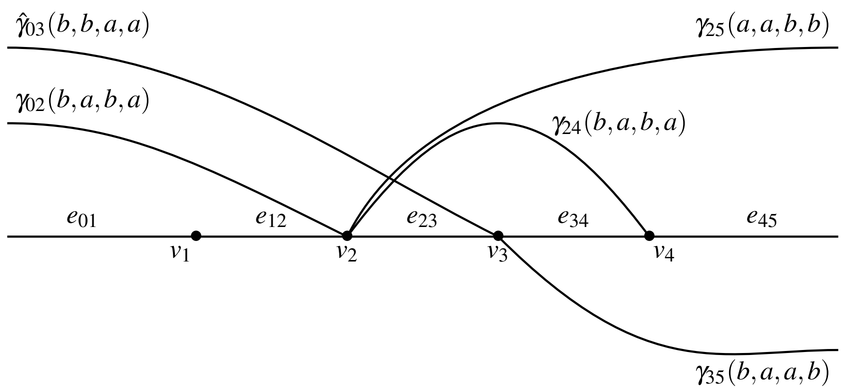
$$f(\sigma^{(a,b,a,b)}(\omega)) = (b-a) + f(\sigma^{(a,b,a,a)}(\omega)). \tag{152}$$

The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_2^C \cap E_3 \cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_3^C \cap \hat{E}_4^C.
\end{aligned} \tag{153}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) = \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \tag{154}$$



Let us denote by $\gamma(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\gamma_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\theta_{25}(a,a,b,b) \in (0, b-a)$ such that the following equality holds

$$T(\gamma_{25}(a,a,b,b), \tilde{\omega}) = e_{23} + e_{34} + e_{45} + 2a + 2\theta_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0, b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b), \tilde{\omega}) = e_{34} + e_{45} + a + \theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) = e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) = e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\hat{\gamma}_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\hat{\theta}_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{03}(b,b,a,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+2a+2\hat{\theta}_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_1 = b$. Let us summarize the conclusions that we can make by analyzing (153). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{24}(b,a,b,a), \theta_{25}(a,a,b,b),$$
$$\theta_{35}(b,a,a,b), \text{ and } \hat{\theta}_{03}(b,b,a,a)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) = \rho+2\theta_{25}(a,a,b,b), \tag{155}$$
$$f(\sigma^{(a,b,b,a)}(\omega)) \geq a-b+f(\sigma^{(a,b,b,b)}(\omega)), \tag{156}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) = \rho+\theta_{02}(b,a,b,a)+\theta_{35}(b,a,a,b), \tag{157}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \tag{158}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho+2\hat{\theta}_{03}(b,b,a,a). \tag{159}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho+2\theta_{25}(a,a,b,b), \tag{160}$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho+\theta_{24}(b,a,b,a), \tag{161}$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq a-b+f(\sigma^{(a,b,a,b)}(\omega)), \tag{162}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a). \tag{163}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a)+2\theta_{25}(a,a,b,b), \tag{164}$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho+2\hat{\theta}_{03}(b,b,a,a)+\theta_{35}(b,a,a,b), \tag{165}$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \tag{166}$$

Combining the equations and inequalities (154–166), we derive

$$\partial_S f(\omega) \geq -2\theta_{25}(a,a,b,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,b,a,b)}(\omega) \in \hat{E}_4$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.6. **Analysis of the case $\sigma^{(a,b,a,b)}(\omega) \in E_3^C$.** The Proposition 3 implies

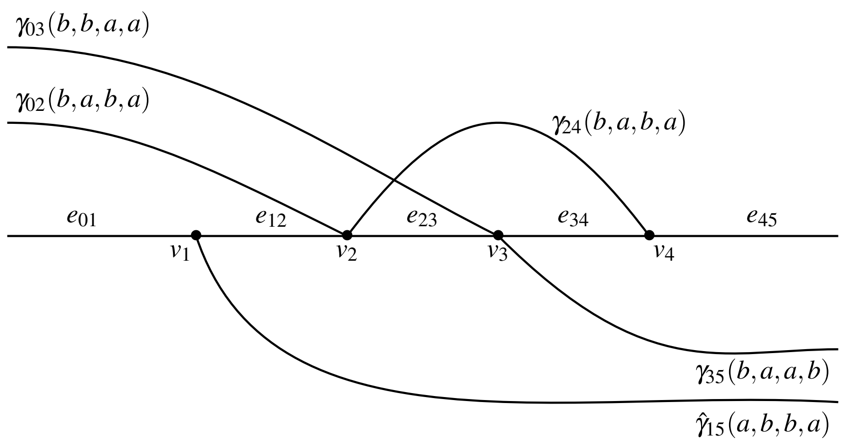
$$f(\sigma^{(a,b,a,b)}(\omega)) = f(\sigma^{(a,b,b,b)}(\omega)). \tag{167}$$

The following relations are satisfied.

$$\begin{array}{rcl} \sigma^{(a,a,a,a)}(\omega) & \in & E_1 \cap E_2 \cap E_3 \cap E_4; \\ \sigma^{(b,b,a,a)}(\omega) & \in & \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4; \\ \sigma^{(b,a,b,a)}(\omega) & \in & \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4; \\ \sigma^{(a,b,b,a)}(\omega) & \in & E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C; \\ \sigma^{(b,a,a,b)}(\omega) & \in & \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C; \\ \sigma^{(a,a,b,b)}(\omega) & \in & E_1 \cap \hat{E}_3^C \cap \hat{E}_4^C. \end{array} \tag{168}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \tag{169}$$



Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{34} + e_{45} + a + \theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{01} + e_{12} + a + \theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) = e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\hat{\gamma}_{15}(a,b,b,a)$ be the section of this geodesic after the edge v_1. There exists a real number $\hat{\theta}_{15}(a,b,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{15}(a,b,b,a),\tilde{\omega}) = e_{12}+e_{23}+e_{34}+e_{45}+3a+2\hat{\theta}_{15}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_4 = a$. Let us summarize the conclusions that we can make by analyzing (168). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{24}(b,a,b,a),$$
$$\theta_{35}(b,a,a,b), \text{ and } \hat{\theta}_{15}(a,b,b,a)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) \geq a-b+f(\sigma^{(b,a,b,b)}(\omega)), \quad (170)$$
$$f(\sigma^{(a,b,b,a)}(\omega)) = \rho+2\hat{\theta}_{15}(a,b,b,a), \quad (171)$$
$$f(\sigma^{(b,a,a,b)}(\omega)) = \rho+\theta_{02}(b,a,b,a)+\theta_{35}(b,a,a,b), \quad (172)$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \quad (173)$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho+2\theta_{03}(b,b,a,a). \quad (174)$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho+\theta_{35}(b,a,a,b), \quad (175)$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho+\theta_{24}(b,a,b,a), \quad (176)$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho+2\hat{\theta}_{15}(a,b,b,a), \quad (177)$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a). \quad (178)$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(a,b,b,b)}(\omega)) \leq f(\sigma^{(a,b,a,b)}(\omega)), \quad (179)$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(b,a,a,b), \quad (180)$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \quad (181)$$

Combining the equations and inequalities (169–181), we derive

$$\partial_S f(\omega) \geq a-b+\theta_{02}(b,a,b,a)-\theta_{35}(b,a,a,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,b,a,b)}(\omega) \in E_3^C$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.7. **Analysis of the case $\sigma^{(a,b,a,b)}(\omega) \in \hat{E}_2$.** The Proposition 3 implies

$$f(\sigma^{(a,b,a,b)}(\omega)) = (b-a) + f(\sigma^{(a,a,a,b)}(\omega)). \tag{182}$$

Initially, we can make the following straightforward conclusions about the environment.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in E_2 \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_4^C.
\end{aligned}$$

We will now prove that the previous equations imply one additional inclusion:

$$\sigma^{(b,a,a,b)}(\omega) \in \hat{E}_1^C.$$

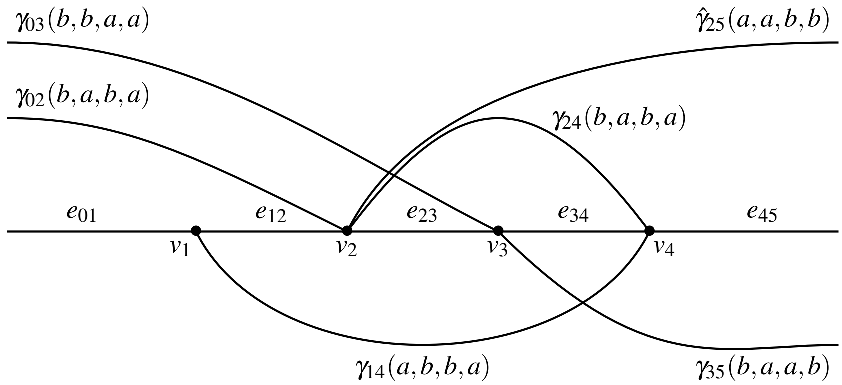
We know that $\sigma^{(b,a,b,a)}(\omega)$ belongs to $\hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C$ and that $\sigma^{(b,a,a,b)}(\omega)$ belongs to $E_2 \cap E_3 \cap \hat{E}_4^C$. Let us consider an arbitrary geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. We know that it must pass through v_2. The section before v_2 has the same passage time as the corresponding section of each geodesic on $\sigma^{(b,a,b,a)}(\omega)$. Hence, each geodesic on $\sigma^{(b,a,a,b)}(\omega)$ must omit v_1.

Hence, we have $\sigma^{(b,a,a,b)}(\omega) \in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C$. The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_4^C.
\end{aligned} \tag{183}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \tag{184}$$



Let us denote by $\gamma(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\gamma_{14}(a,b,b,a)$ be the section of this geodesic between the edges v_1 and v_4. There exists a real number $\theta_{14}(a,b,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{14}(a,b,b,a),\tilde{\omega}) \quad = \quad e_{12}+e_{23}+e_{34}+2a+2\theta_{14}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{34}+e_{45}+a+\theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\hat{\gamma}_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\hat{\theta}_{25}(a,a,b,b) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{25}(a,a,b,b),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+e_{45}+2a+2\hat{\theta}_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_3 = b$. Let us summarize the conclusions that we can make by analyzing (183). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{14}(a,b,b,a),$$
$$\theta_{24}(b,a,b,a), \theta_{35}(b,a,a,b), \text{ and } \hat{\theta}_{25}(a,a,b,b)$$

that all belong to the interval $(0, b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) \geq a - b + f(\sigma^{(a,b,b,b)}(\omega)), \quad (185)$$
$$f(\sigma^{(a,b,b,a)}(\omega)) = \rho + 2\theta_{14}(a,b,b,a), \quad (186)$$
$$f(\sigma^{(b,a,a,b)}(\omega)) = \rho + \theta_{02}(b,a,b,a) + \theta_{35}(b,a,a,b), \quad (187)$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho + \theta_{02}(b,a,b,a) + \theta_{24}(b,a,b,a), \quad (188)$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho + 2\theta_{03}(b,b,a,a). \quad (189)$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq a - b + f(\sigma^{(a,b,a,b)}(\omega)), \quad (190)$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho + \theta_{24}(b,a,b,a), \quad (191)$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho + 2\theta_{14}(a,b,b,a), \quad (192)$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho + \theta_{02}(b,a,b,a). \quad (193)$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho + \theta_{02}(b,a,b,a) + 2\hat{\theta}_{25}(a,a,b,b), \quad (194)$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho + 2\theta_{03}(b,b,a,a) + \theta_{35}(b,a,a,b), \quad (195)$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \quad (196)$$

Combining the equations and inequalities (184–196), we derive

$$\partial_S f(\omega) \geq -2\hat{\theta}_{25}(a,a,b,b).$$

Recall that each of the constants θ belongs to the interval $[0, b-a]$. Thus, under the condition $\sigma^{(a,b,a,b)}(\omega) \in \hat{E}_2$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.8. **Analysis of the case $\sigma^{(a,b,a,b)}(\omega) \in E_1^C$.** The Proposition 3 implies

$$f(\sigma^{(a,b,a,b)}(\omega)) = f(\sigma^{(b,b,a,b)}(\omega)). \quad (197)$$

Initially, we can make the following straightforward conclusions about the environment.

$$\sigma^{(a,a,a,a)}(\omega) \in E_1 \cap E_2 \cap E_3 \cap E_4;$$
$$\sigma^{(b,b,a,a)}(\omega) \in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3;$$
$$\sigma^{(b,a,b,a)}(\omega) \in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4;$$
$$\sigma^{(a,b,b,a)}(\omega) \in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4;$$
$$\sigma^{(b,a,a,b)}(\omega) \in \hat{E}_1^C \cap E_3 \cap \hat{E}_4^C;$$
$$\sigma^{(a,a,b,b)}(\omega) \in E_1 \cap E_2 \cap \hat{E}_3^C \cap \hat{E}_4^C.$$

We will now prove that the previous equations imply one additional inclusion:

$$\sigma^{(b,b,a,a)}(\omega) \in E_4.$$

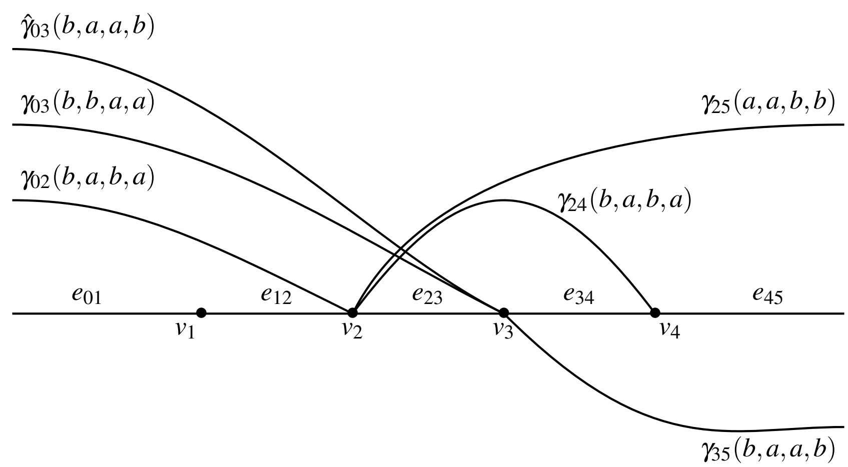
We know that $\sigma^{(a,a,a,a)}(\omega)$ belongs to $E_1 \cap E_2 \cap E_3 \cap E_4$ and that $\sigma^{(b,b,a,a)}(\omega)$ belongs to $\hat{E}_1^C \cap \hat{E}_2^C \cap E_3$. Let us consider an arbitrary geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. We know that it must pass through v_3. The section after v_3 has the same passage time as the corresponding section of each geodesic on $\sigma^{(a,a,a,a)}(\omega)$. Hence, each geodesic on $\sigma^{(b,b,a,a)}(\omega)$ must pass through v_4.

Hence, we have $\sigma^{(b,b,a,a)}(\omega) \in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4$. The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4; \\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4; \\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4; \\
\sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4; \\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_3 \cap \hat{E}_4^C; \\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_3^C \cap \hat{E}_4^C. \qquad (198)
\end{aligned}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \qquad (199)$$



Let us denote by $\gamma(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\gamma_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\theta_{25}(a,a,b,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{25}(a,a,b,b),\tilde{\omega}) \quad = \quad e_{23} + e_{34} + e_{45} + 2a + 2\theta_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{34} + e_{45} + a + \theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\hat{\gamma}_{03}(b,a,a,b)$ be the section of this geodesic before the edge v_3. There exists a real number $\hat{\theta}_{03}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{03}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+e_{23}+2a+\hat{\theta}_{03}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_2 = a$. Let us summarize the conclusions that we can make by analyzing (198). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{24}(b,a,b,a),$$
$$\theta_{25}(a,a,b,b), \theta_{35}(b,a,a,b), \text{ and } \hat{\theta}_{03}(b,a,a,b)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) \quad = \quad \rho+2\theta_{25}(a,a,b,b), \tag{200}$$
$$f(\sigma^{(a,b,b,a)}(\omega)) \quad \geq \quad a-b+f(\sigma^{(a,b,b,b)}(\omega)), \tag{201}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) \quad = \quad \rho+\hat{\theta}_{03}(b,a,a,b)+\theta_{35}(b,a,a,b), \tag{202}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) \quad = \quad \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \tag{203}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) \quad = \quad \rho+2\theta_{03}(b,b,a,a). \tag{204}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \quad \leq \quad \rho+\theta_{35}(b,a,a,b), \tag{205}$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \quad \leq \quad \rho+\theta_{24}(b,a,b,a), \tag{206}$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \quad \leq \quad \rho+2\theta_{03}(b,b,a,a), \tag{207}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \quad \leq \quad \rho+\theta_{02}(b,a,b,a). \tag{208}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$\begin{aligned}
f(\sigma^{(b,a,b,b)}(\omega)) &\leq \rho+\theta_{02}(b,a,b,a)+2\theta_{25}(a,a,b,b), &(209)\\
f(\sigma^{(b,b,a,b)}(\omega)) &\leq f(\sigma^{(a,b,a,b)}(\omega)), &(210)\\
f(\sigma^{(b,b,b,a)}(\omega)) &\leq f(\sigma^{(b,b,b,b)}(\omega)). &(211)
\end{aligned}$$

Combining the equations and inequalities (199–211), we derive

$$\partial_S f(\omega) \quad \geq \quad a-b-\theta_{02}(b,a,b,a)+\hat{\theta}_{03}(b,a,a,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,b,a,b)}(\omega)\in E_1^C$, we have obtained that $\partial_S f(\omega)\geq -2(b-a)$.

6.9. **Analysis of the case $\sigma^{(b,a,a,b)}(\omega)\in E_2^C$.** The Proposition 3 implies

$$f(\sigma^{(b,a,a,b)}(\omega)) = f(\sigma^{(b,b,a,b)}(\omega)). \tag{212}$$

Initially, we can make the following straightforward conclusions about the environment.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap \hat{E}_2^C\cap E_3;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap E_2\cap \hat{E}_3^C\cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1\cap \hat{E}_2^C\cap \hat{E}_3^C\cap E_4;\\
\sigma^{(a,b,a,b)}(\omega) &\in \hat{E}_2^C\cap E_3\cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1\cap E_2\cap \hat{E}_3^C\cap \hat{E}_4^C.
\end{aligned}$$

We will now prove that the previous equations imply one additional inclusion:

$$\sigma^{(b,b,a,a)}(\omega)\in E_4.$$

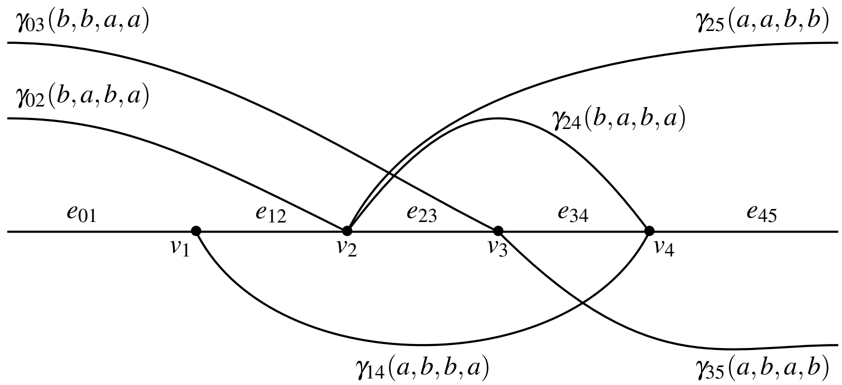
We know that $\sigma^{(a,a,a,a)}(\omega)$ belongs to $E_1\cap E_2\cap E_3\cap E_4$ and that $\sigma^{(b,b,a,a)}(\omega)$ belongs to $\hat{E}_1^C\cap\hat{E}_2^C\cap E_3$. Let us consider an arbitrary geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. We know that it must pass through v_3. The section after v_3 has the same passage time as the corresponding section of each geodesic on $\sigma^{(a,a,a,a)}(\omega)$. Hence, each geodesic on $\sigma^{(b,b,a,a)}(\omega)$ must pass through v_4.

Hence, we have $\sigma^{(b,b,a,a)}(\omega)\in\hat{E}_1^C\cap\hat{E}_2^C\cap E_3\cap E_4$. The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap \hat{E}_2^C\cap E_3\cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap E_2\cap \hat{E}_3^C\cap E_4;\\
\sigma^{(a,b,b,a)}(\omega) &\in E_1\cap \hat{E}_2^C\cap \hat{E}_3^C\cap E_4;\\
\sigma^{(a,b,a,b)}(\omega) &\in \hat{E}_2^C\cap E_3\cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1\cap E_2\cap \hat{E}_3^C\cap \hat{E}_4^C. &(213)
\end{aligned}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i\in\{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i\in\{1,2,3\}$ and $j=i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho, \text{ where } \rho = 4a+e_{01}+e_{12}+e_{23}+e_{34}+e_{45}. \tag{214}$$



Let us denote by $\gamma(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\gamma_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\theta_{25}(a,a,b,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{25}(a,a,b,b),\tilde{\omega}) = e_{23}+e_{34}+e_{45}+2a+2\theta_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(a,b,a,b)$ a geodesic on the environment $\sigma^{(a,b,a,b)}(\omega)$. Let $\gamma_{35}(a,b,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(a,b,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(a,b,a,b),\tilde{\omega}) = e_{34}+e_{45}+a+\theta_{35}(a,b,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\gamma_{14}(a,b,b,a)$ be the section of this geodesic between the edges v_1 and v_4. There exists a real number $\theta_{14}(a,b,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{14}(a,b,b,a),\tilde{\omega}) = e_{12}+e_{23}+e_{34}+2a+2\theta_{14}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) = e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) = e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us summarize the conclusions that we can make by analyzing (213). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{14}(a,b,b,a),$$
$$\theta_{24}(b,a,b,a), \theta_{25}(a,a,b,b), \text{ and } \theta_{35}(a,b,a,b)$$

that all belong to the interval $(0, b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) = \rho + 2\theta_{25}(a,a,b,b), \quad (215)$$
$$f(\sigma^{(a,b,a,b)}(\omega)) \geq a - b + f(\sigma^{(a,b,b,b)}(\omega)), \quad (216)$$
$$f(\sigma^{(a,b,b,a)}(\omega)) = \rho + 2\theta_{14}(a,b,b,a), \quad (217)$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho + \theta_{02}(b,a,b,a) + \theta_{24}(b,a,b,a), \quad (218)$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho + 2\theta_{03}(b,b,a,a). \quad (219)$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho + \theta_{35}(a,b,a,b), \quad (220)$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho + \theta_{24}(b,a,b,a), \quad (221)$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho + 2\theta_{14}(a,b,b,a), \quad (222)$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho + 2\theta_{03}(b,b,a,a). \quad (223)$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho + \theta_{02}(b,a,b,a) + 2\theta_{25}(a,a,b,b), \quad (224)$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq f(\sigma^{(b,a,a,b)}(\omega)), \quad (225)$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \quad (226)$$

Combining the equations and inequalities (214–226), we derive

$$\partial_S f(\omega) \geq a - b - \theta_{35}(a,b,a,b).$$

Recall that each of the constants θ belongs to the interval $[0, b-a]$. Thus, under the condition $\sigma^{(b,a,a,b)}(\omega) \in E_2^C$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.10. **Analysis of the case $\sigma^{(b,a,a,b)}(\omega) \in \hat{E}_1$.** The Proposition 3 implies

$$f(\sigma^{(b,a,a,b)}(\omega)) = (b-a) + f(\sigma^{(a,a,a,b)}(\omega)). \quad (227)$$

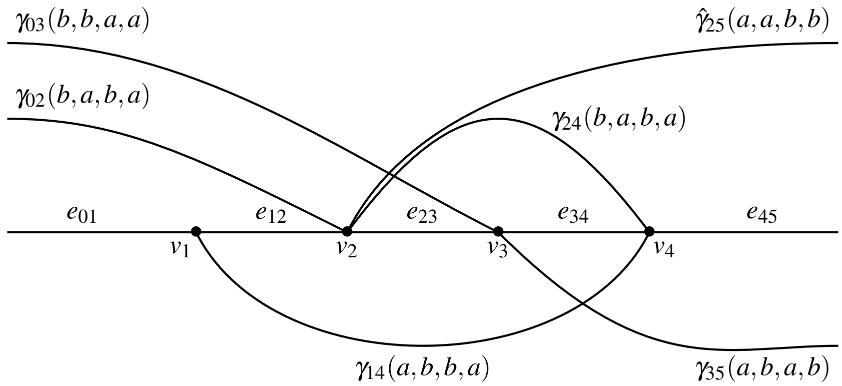
The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4; \\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4; \\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4; \\
\sigma^{(a,b,b,a)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap \hat{E}_3^C \cap E_4; \\
\sigma^{(a,b,a,b)}(\omega) &\in E_1 \cap E_3 \cap \hat{E}_4^C; \\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_4^C.
\end{aligned} \quad (228)$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over

$\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \qquad (229)$$



Let us denote by $\gamma(a,b,a,b)$ a geodesic on the environment $\sigma^{(a,b,a,b)}(\omega)$. Let $\gamma_{35}(a,b,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(a,b,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(a,b,a,b),\tilde{\omega}) \quad = \quad e_{34} + e_{45} + a + \theta_{35}(a,b,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(a,b,b,a)$ a geodesic on the environment $\sigma^{(a,b,b,a)}(\omega)$. Let $\gamma_{14}(a,b,b,a)$ be the section of this geodesic between the edges v_1 and v_4. There exists a real number $\theta_{14}(a,b,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{14}(a,b,b,a),\tilde{\omega}) \quad = \quad e_{12} + e_{23} + e_{34} + 2a + 2\theta_{14}(a,b,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{01} + e_{12} + a + \theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{23} + e_{34} + a + \theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) \quad = \quad e_{01} + e_{12} + e_{23} + 2a + 2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\hat{\gamma}_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\hat{\theta}_{25}(a,a,b,b) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{25}(a,a,b,b),\tilde{\omega}) = e_{23}+e_{34}+e_{45}+2a+2\hat{\theta}_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_3 = b$. Let us summarize the conclusions that we can make by analyzing (228). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{14}(a,b,b,a),$$
$$\theta_{24}(b,a,b,a), \theta_{35}(a,b,a,b), \text{ and } \hat{\theta}_{25}(a,a,b,b)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) = \rho+2\hat{\theta}_{25}(a,a,b,b), \quad (230)$$
$$f(\sigma^{(a,b,a,b)}(\omega)) \geq a-b+f(\sigma^{(a,b,b,b)}(\omega)), \quad (231)$$
$$f(\sigma^{(a,b,b,a)}(\omega)) = \rho+2\theta_{14}(a,b,b,a), \quad (232)$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \quad (233)$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho+2\theta_{03}(b,b,a,a). \quad (234)$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq a-b+f(\sigma^{(b,a,a,b)}(\omega)), \quad (235)$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho+\theta_{24}(b,a,b,a), \quad (236)$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho+2\theta_{14}(a,b,b,a), \quad (237)$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a). \quad (238)$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a)+2\hat{\theta}_{25}(a,a,b,b), \quad (239)$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(a,b,a,b), \quad (240)$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \quad (241)$$

Combining the equations and inequalities (229–241), we derive

$$\partial_S f(\omega) \geq -\theta_{02}(b,a,b,a)-\theta_{35}(a,b,a,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(b,a,a,b)}(\omega) \in \hat{E}_1$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

6.11. **Analysis of the case $\sigma^{(a,b,b,a)}(\omega) \in \hat{E}_2$.** The Proposition 3 implies

$$f(\sigma^{(a,b,b,a)}(\omega)) = (b-a)+f(\sigma^{(a,a,b,a)}(\omega)). \quad (242)$$

Initially, we can make the following straightforward conclusions about the environment.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in E_2 \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,b,a,b)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_3^C.
\end{aligned}$$

We will now prove that the previous equations imply one additional inclusion:

$$\sigma^{(b,a,b,a)}(\omega) \in \hat{E}_1^C.$$

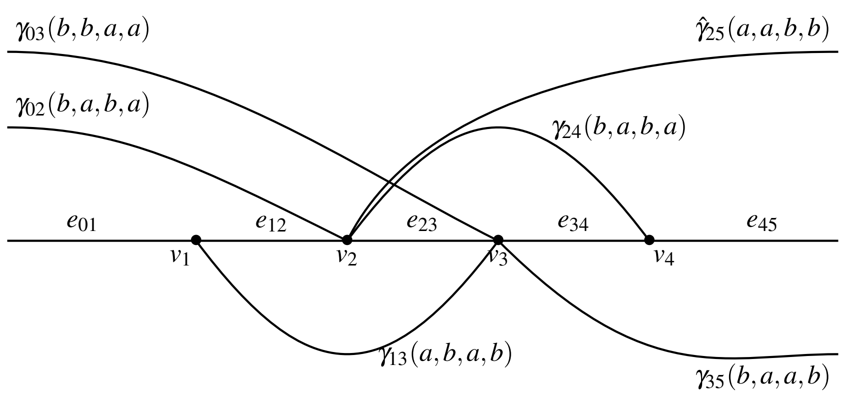
We know that $\sigma^{(b,a,a,b)}(\omega)$ belongs to $\hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C$ and that $\sigma^{(b,a,b,a)}(\omega)$ belongs to $E_2 \cap \hat{E}_3^C \cap E_4$. Let us consider an arbitrary geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. We know that it must pass through v_2. The section before v_2 has the same passage time as the corresponding section of each geodesic on $\sigma^{(b,a,a,b)}(\omega)$. Hence, each geodesic on $\sigma^{(b,a,b,a)}(\omega)$ must omit v_1.

Hence, we have $\sigma^{(b,a,b,a)}(\omega) \in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4$. The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1 \cap E_2 \cap E_3 \cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C \cap \hat{E}_2^C \cap E_3 \cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap \hat{E}_3^C \cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C \cap E_2 \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,b,a,b)}(\omega) &\in E_1 \cap \hat{E}_2^C \cap E_3 \cap \hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1 \cap E_2 \cap \hat{E}_3^C. \qquad (243)
\end{aligned}$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i \in \{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i \in \{1,2,3\}$ and $j = i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) \quad = \quad \rho, \text{ where } \rho = 4a + e_{01} + e_{12} + e_{23} + e_{34} + e_{45}. \qquad (244)$$



Let us denote by $\gamma(a,b,a,b)$ a geodesic on the environment $\sigma^{(a,b,a,b)}(\omega)$. Let $\gamma_{13}(a,b,a,b)$ be the section of this geodesic between the edges v_1 and v_3. There exists a real number $\theta_{13}(a,b,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{13}(a,b,a,b),\tilde{\omega}) \quad = \quad e_{12}+e_{23}+a+\theta_{13}(a,b,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) \quad = \quad e_{34}+e_{45}+a+\theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{02}(b,a,b,a)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+a+\theta_{02}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\gamma_{24}(b,a,b,a)$ be the section of this geodesic between the edges v_2 and v_4. There exists a real number $\theta_{24}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{24}(b,a,b,a),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+a+\theta_{24}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\gamma_{03}(b,b,a,a)$ be the section of this geodesic before the edge v_3. There exists a real number $\theta_{03}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{03}(b,b,a,a),\tilde{\omega}) \quad = \quad e_{01}+e_{12}+e_{23}+2a+2\theta_{03}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\hat{\gamma}_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\hat{\theta}_{25}(a,a,b,b) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{25}(a,a,b,b),\tilde{\omega}) \quad = \quad e_{23}+e_{34}+e_{45}+2a+2\hat{\theta}_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_4 = b$. Let us summarize the conclusions that we can make by analyzing (243). We obtained that there exist scalars

$$\theta_{02}(b,a,b,a), \theta_{03}(b,b,a,a), \theta_{13}(a,b,a,b),$$
$$\theta_{24}(b,a,b,a), \theta_{35}(b,a,a,b), \text{ and } \hat{\theta}_{25}(a,a,b,b)$$

that all belong to the interval $(0,b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) \quad \geq \quad a-b+f(\sigma^{(a,b,b,b)}(\omega)), \tag{245}$$
$$f(\sigma^{(a,b,a,b)}(\omega)) \quad = \quad \rho+\theta_{13}(a,b,a,b)+\theta_{35}(b,a,a,b), \tag{246}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) \quad = \quad \rho+\theta_{02}(b,a,b,a)+\theta_{35}(b,a,a,b), \tag{247}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) \quad = \quad \rho+\theta_{02}(b,a,b,a)+\theta_{24}(b,a,b,a), \tag{248}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) \quad = \quad \rho+2\theta_{03}(b,b,a,a). \tag{249}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho+\theta_{35}(b,a,a,b), \quad (250)$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq a-b+f(\sigma^{(a,b,b,a)}(\omega)), \quad (251)$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho+\theta_{13}(a,b,a,b), \quad (252)$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a). \quad (253)$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho+\theta_{02}(b,a,b,a)+2\hat{\theta}_{25}(a,a,b,b), \quad (254)$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq \rho+2\theta_{03}(b,b,a,a)+\theta_{35}(b,a,a,b), \quad (255)$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)). \quad (256)$$

Combining the equations and inequalities (244–256), we derive

$$\partial_S f(\omega) \geq \theta_{24}(b,a,b,a)-2\hat{\theta}_{25}(a,a,b,b).$$

Recall that each of the constants θ belongs to the interval $[0,b-a]$. Thus, under the condition $\sigma^{(a,b,b,a)}(\omega)\in\hat{E}_2$, we have obtained that $\partial_S f(\omega)\geq -2(b-a)$.

6.12. **Analysis of the case $\sigma^{(a,b,b,a)}(\omega)\in E_1^C$.** The Proposition 3 implies

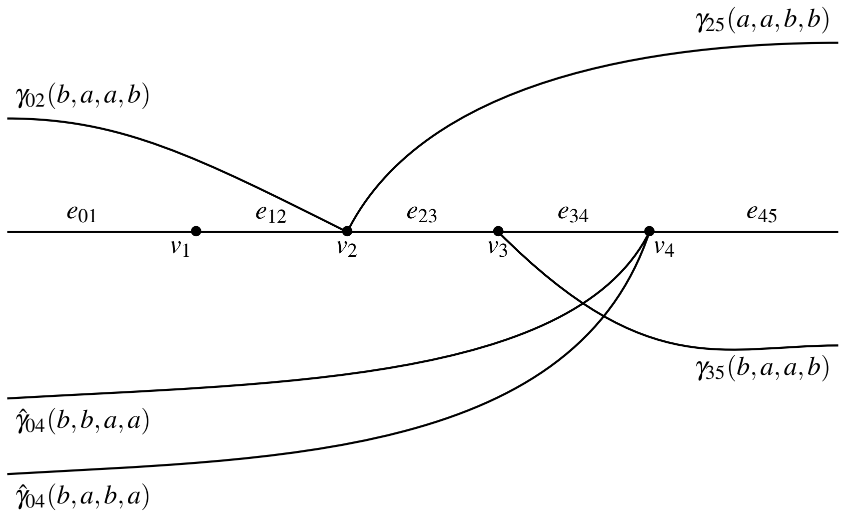
$$f(\sigma^{(a,b,b,a)}(\omega))=f(\sigma^{(b,b,b,a)}(\omega)). \quad (257)$$

The following relations are satisfied.

$$\begin{aligned}
\sigma^{(a,a,a,a)}(\omega) &\in E_1\cap E_2\cap E_3\cap E_4;\\
\sigma^{(b,b,a,a)}(\omega) &\in \hat{E}_1^C\cap\hat{E}_2^C\cap E_4;\\
\sigma^{(b,a,b,a)}(\omega) &\in \hat{E}_1^C\cap\hat{E}_3^C\cap E_4;\\
\sigma^{(b,a,a,b)}(\omega) &\in \hat{E}_1^C\cap E_2\cap E_3\cap\hat{E}_4^C;\\
\sigma^{(a,b,a,b)}(\omega) &\in E_1\cap\hat{E}_2^C\cap E_3\cap\hat{E}_4^C;\\
\sigma^{(a,a,b,b)}(\omega) &\in E_1\cap E_2\cap\hat{E}_3^C\cap\hat{E}_4^C.
\end{aligned} \quad (258)$$

Denote by $\gamma(a,a,a,a)$ the geodesic on the environment $\sigma^{(a,a,a,a)}(\omega)$. Let $\gamma^-(a,a,a,a)$ be the section of $\gamma(a,a,a,a)$ before v_1. Let $\gamma^+(a,a,a,a)$ be the section after v_4. Let $\gamma(i,i+1)$ be the section between v_i and v_{i+1} for $i\in\{1,2,3,4\}$. Let e_{01} be the passage time over $\gamma^-(a,a,a,a)$. Let e_{45} be the passage time over $\gamma^+(a,a,a,a)$. For $i\in\{1,2,3\}$ and $j=i+1$, denote by e_{ij} the passage time over $\gamma(i,j)$. The value $f(\sigma^{(a,a,a,a)}(\omega))$ satisfies

$$f(\sigma^{(a,a,a,a)}(\omega)) = \rho, \text{ where } \rho=4a+e_{01}+e_{12}+e_{23}+e_{34}+e_{45}. \quad (259)$$



Let us denote by $\gamma(a,a,b,b)$ a geodesic on the environment $\sigma^{(a,a,b,b)}(\omega)$. Let $\gamma_{25}(a,a,b,b)$ be the section of this geodesic after the edge v_2. There exists a real number $\theta_{25}(a,a,b,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{25}(a,a,b,b),\tilde{\omega}) = e_{23}+e_{34}+e_{45}+2a+2\theta_{25}(a,a,b,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{02}(b,a,a,b)$ be the section of this geodesic before the edge v_2. There exists a real number $\theta_{02}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{02}(b,a,a,b),\tilde{\omega}) = e_{01}+e_{12}+a+\theta_{02}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\gamma(b,a,a,b)$ a geodesic on the environment $\sigma^{(b,a,a,b)}(\omega)$. Let $\gamma_{35}(b,a,a,b)$ be the section of this geodesic after the edge v_3. There exists a real number $\theta_{35}(b,a,a,b) \in (0,b-a)$ such that the following equality holds

$$T(\gamma_{35}(b,a,a,b),\tilde{\omega}) = e_{34}+e_{45}+a+\theta_{35}(b,a,a,b).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n. Let us denote by $\hat{\gamma}(b,a,b,a)$ a geodesic on the environment $\sigma^{(b,a,b,a)}(\omega)$. Let $\hat{\gamma}_{04}(b,a,b,a)$ be the section of this geodesic before the edge v_4. There exists a real number $\hat{\theta}_{04}(b,a,b,a) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{04}(b,a,b,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+e_{34}+3a+2\hat{\theta}_{04}(b,a,b,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_2 = a$. Let us denote by $\hat{\gamma}(b,b,a,a)$ a geodesic on the environment $\sigma^{(b,b,a,a)}(\omega)$. Let $\hat{\gamma}_{04}(b,b,a,a)$ be the section of this geodesic before the edge v_4. There exists a real number $\hat{\theta}_{04}(b,b,a,a) \in (0,b-a)$ such that the following equality holds

$$T(\hat{\gamma}_{04}(b,b,a,a),\tilde{\omega}) = e_{01}+e_{12}+e_{23}+e_{34}+3a+2\hat{\theta}_{04}(b,b,a,a).$$

The outcome $\tilde{\omega}$ in the previous equation can be any element of Ω_n that satisfies $\tilde{\omega}_3 = a$. Let us summarize the conclusions that we can make by analyzing (258). We obtained that there exist scalars

$$\theta_{02}(b,a,a,b), \theta_{25}(a,a,b,b), \theta_{35}(b,a,a,b),$$
$$\hat{\theta}_{04}(b,a,b,a), \text{ and } \hat{\theta}_{04}(b,b,a,a)$$

that all belong to the interval $(0, b-a)$ for which the following relations hold

$$f(\sigma^{(a,a,b,b)}(\omega)) = \rho + 2\theta_{25}(a,a,b,b), \tag{260}$$
$$f(\sigma^{(a,b,a,b)}(\omega)) \geq a - b + f(\sigma^{(a,b,b,b)}(\omega)), \tag{261}$$
$$f(\sigma^{(b,a,a,b)}(\omega)) = \rho + \theta_{02}(b,a,a,b) + \theta_{35}(b,a,a,b), \tag{262}$$
$$f(\sigma^{(b,a,b,a)}(\omega)) = \rho + 2\hat{\theta}_{04}(b,a,b,a), \tag{263}$$
$$f(\sigma^{(b,b,a,a)}(\omega)) = \rho + 2\hat{\theta}_{04}(b,b,a,a). \tag{264}$$

The following inequalities are derived by identifying appropriate segments and calculating their passage times.

$$f(\sigma^{(a,a,a,b)}(\omega)) \leq \rho + \theta_{35}(b,a,a,b), \tag{265}$$
$$f(\sigma^{(a,a,b,a)}(\omega)) \leq \rho + 2\hat{\theta}_{04}(b,a,b,a), \tag{266}$$
$$f(\sigma^{(a,b,a,a)}(\omega)) \leq \rho + 2\hat{\theta}_{04}(b,b,a,a), \tag{267}$$
$$f(\sigma^{(b,a,a,a)}(\omega)) \leq \rho + \theta_{02}(b,a,a,b). \tag{268}$$

In a similar way we derive inequalities for the environment shifts that contain three operators of the type σ^b.

$$f(\sigma^{(b,a,b,b)}(\omega)) \leq \rho + \theta_{02}(b,a,a,b) + 2\theta_{25}(a,a,b,b), \tag{269}$$
$$f(\sigma^{(b,b,a,b)}(\omega)) \leq f(\sigma^{(b,b,b,b)}(\omega)), \tag{270}$$
$$f(\sigma^{(b,b,b,a)}(\omega)) \leq f(\sigma^{(a,b,b,a)}(\omega)). \tag{271}$$

Combining the equations and inequalities (259–271), we derive

$$\partial_S f(\omega) \geq a - b - \theta_{02}(b,a,a,b).$$

Recall that each of the constants θ belongs to the interval $[0, b-a]$. Thus, under the condition $\sigma^{(a,b,b,a)}(\omega) \in E_1^C$, we have obtained that $\partial_S f(\omega) \geq -2(b-a)$.

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