ALMOST SURE BOUNDS FOR HIGHER-ORDER DERIVATIVES OF FIRST-PASSAGE PERCOLATION WITH RESPECT TO THE ENVIRONMENT

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ABSTRACT. The variance of the first-passage percolation is bounded by the L^2-norms of the derivatives with respect to the environment. In this paper we prove that the environment derivatives of orders $k \in \{2,3,4\}$ are bounded below by $-\binom{k-2}{\lceil \frac{k-2}{2} \rceil}$ and above by $\binom{k-2}{\lceil \frac{k-2}{2} \rceil}$. We believe that these are the bounds for all k. We provide examples of environments that show that these extreme values can be attained.

1. INTRODUCTION

1.1. **Model.** We will use the same model and the definitions as in [4]. We refer the reader to the same paper for more comprehensive historical remarks and overview of the literature. For completeness, we will define the model here and state the results that we will need for the proofs.

Let $a < b$ be two fixed positive real numbers, and let $p \in (0,1)$ be a fixed probability. We consider a finite subgraph of the integer lattice $\mathbb{Z}^d$ (for $d \geq 2$), restricted to the box $[-2n, 2n]^d$. Two vertices $(x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ are connected by an edge if they are nearest neighbors, i.e., if $|x_1 - y_1| + \cdots + |x_d - y_d| = 1$.

Each edge e in this graph is independently assigned a random passage time, taking the value a with probability p and b with probability $1-p$. Let W_n denote the set of all such edges, and define the sample space as $\Omega_n = \{a,b\}^{W_n}$, where each element $\omega \in \Omega_n$ specifies a particular realization of passage times on the edges.

Given a configuration ω and a path γ (a sequence of adjacent edges), the passage time $T(\gamma, \omega)$ is defined as the sum of the passage times assigned to the edges in γ. For any two vertices u and v, the function $f(u, v, \omega)$ denotes the minimal passage time over all paths connecting u to v in the configuration ω.

When the destination vertex v is fixed, we may write $f_n(\omega)$, or simply f_n, to refer to $f(0, nv, \omega)$.

A path γ is called *geodesic* if the minimum $f_n(\omega)$ is attained at γ, i.e. if $f_n(\omega) = T(\gamma, \omega)$.

1.2. **Environment derivatives.** If we denote by W_n the set of all edges, then the sample space is $\Omega_n = \{a,b\}^{W_n}$. We will often omit the subscript n, when there is no danger of confusion. For each edge j and each $\omega \in \Omega$, we define $\sigma_j^a(\omega)$ as the element of Ω whose j-th coordinate is changed from ω_j to a, regardless of what the original value ω_j was. The operation σ_j^b is defined in analogous way. Formally, for $\delta \in \{a,b\}$, we define $\sigma_j^\delta : \Omega \to \Omega$ with

$$\left[\sigma_j^\delta(\omega)\right]_k = \begin{cases} \omega_k, & k \neq j, \\ \delta, & k = j. \end{cases} \qquad (1)$$

If $\varphi : \Omega \to \mathbb{R}$ is any random variable, then the *first order environment derivative* $\partial_j \varphi$ is the random variable defined as

$$\partial_j \varphi \quad = \quad \varphi \circ \sigma_j^b - \varphi \circ \sigma_j^a. \tag{2}$$

For two distinct vertices k and l, we will give the name *second order environment derivative* to the quantity $\partial_k \partial_l \varphi$. In general, if S is a non-empty subset of W, the operator $\partial_S \varphi$ is defined recursively as

$$\partial_S \varphi \quad = \quad \partial_{S \setminus \{j\}} (\partial_j \varphi), \tag{3}$$

where j is an arbitrary element of S. The definition (3) is independent on the choice of j, since a simple induction can be used to prove that for $S = \{s_1, \dots, s_m\}$, the following holds

$$\partial_S \varphi \quad = \quad \sum_{\theta_1 \in \{a,b\}} \cdots \sum_{\theta_m \in \{a,b\}} (-1)^{\mathbf{1}_a(\theta_1) + \cdots + \mathbf{1}_a(\theta_m)} \varphi \circ \sigma_{s_1}^{\theta_1} \circ \cdots \circ \sigma_{s_m}^{\theta_m}. \tag{4}$$

The function $\mathbf{1}_a : \{a, b\} \to \{0, 1\}$ in (4) assigns the value 1 to a and 0 to b.

1.3. **Bounds on variance.** A thorough understanding of environment derivatives would lead to a complete understanding of the variance. As shown in [4], the variance can be decomposed as

$$\operatorname{var}(f) \quad = \quad \sum_{M \subseteq W, M \neq \emptyset} (p(1-p))^{|M|} \left(\mathbb{E}\left[\partial_M f \right] \right)^2,$$

which highlights the central role of the L^2-norms of the derivatives $\partial_M f$ in determining fluctuation behavior. Despite their importance, general bounds for these norms are not yet available.

The following result from [4] provides a bound on the variance in terms of environment derivatives which generalizes results from [5] and [6].

Theorem 1. *Let f be a random variable on Ω. For every integer $k \geq 1$, there exists a real constant C and an integer n_0 such that for $n \geq n_0$, the following inequality holds*

$$\begin{aligned} var(f) \quad \leq & \sum_{M \subseteq W, 1 \leq |M| < k} (p(1-p))^{|M|} \left(\mathbb{E}\left[\partial_M f \right] \right)^2 \\ & + C \cdot \sum_{M \subseteq W, |M| = k, \|\partial_M f\|_1 \neq 0} \frac{\|\partial_M f\|_2^2}{1 + \left(\log \frac{\|\partial_M f\|_2}{\|\partial_M f\|_1} \right)^k}, \end{aligned} \tag{5}$$

where $\|g\|_p$ is the L^p-norm of the function g defined as

$$\|g\|_p = \left(\int_\Omega |g|^p \, d\mathbb{P} \right)^{1/p} = \left(\mathbb{E}[|g|^p] \right)^{1/p}.$$

A widely conjectured upper bound for the variance in first-passage percolation is $C \cdot n^{2\chi}$, where the exponent χ depends on the dimension d. In two dimensions, current heuristics and numerical evidence suggest that $\chi = \frac{1}{3}$ [1]. A rigorous upper bound of $\chi \leq \frac{1}{2}$ was proven by Kesten in 1993 [3], but a formal proof that χ is strictly less than $\frac{1}{2}$ remains elusive. For dimensions $d > 2$, even conjectural values of χ are not firmly established. Nevertheless, as noted in [1], it is widely believed that χ remains positive in all dimensions, with its value tending to zero as the dimension increases.

The best known upper bound on the variance is currently $C \cdot \frac{n}{\log n}$, due to the work of Benjamini, Kalai, and Schramm [2]. Their argument relies on Talagrand's inequality [5], which involves first-order environment derivatives.

We conjecture that the L^2-norms $\|\partial_M f\|_2$ are small, particularly in dimensions $d \geq 3$, where exponential decay may occur. However, at present, these quantities remain analytically difficult to control.

This paper contributes to the analysis of environment derivatives by completing the picture of almost sure bounds for all derivative orders up to four.

1.4. **Almost sure bounds on environment derivatives.** The sequence $(\mathscr{U}_1, \mathscr{U}_2, \dots)$ represents the most optimal upper bounds for environment derivatives. The number $\mathscr{U}_k$ is defined as the best upper bound on the k-th order environment derivative, i.e.

$$\mathscr{U}_k = \frac{1}{b-a} \max\left\{\partial_S f_n(\omega) : n \in \mathbb{N}, S \subseteq W_n, |S| = k, \omega \in \Omega_n\right\}. \tag{6}$$

The sequence $(\mathscr{L}_1, \mathscr{L}_2, \dots)$ of the most optimal lower bounds is defined in an analogous way

$$\mathscr{L}_k = \frac{1}{b-a} \min\left\{\partial_S f_n(\omega) : n \in \mathbb{N}, S \subseteq W_n, |S| = k, \omega \in \Omega_n\right\}. \tag{7}$$

Theorem 2. *In dimensions $d \geq 3$, the first four values of $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ are shown in the table below.*

k	1	2	3	4
$\mathscr{U}_k$	1	1	1	2
$\mathscr{L}_k$	0	-1	-1	-2

(8)

Theorem 3. *The sequences $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ satisfy*

$$\mathscr{U}_{k+1} \leq \mathscr{U}_k - \mathscr{L}_k \quad \textit{and} \quad \mathscr{L}_{k+1} \geq \mathscr{L}_k - \mathscr{U}_k, \tag{9}$$

for all $k \geq 1$. Moreover, for all $k \geq 2$,

$$\mathscr{U}_k \leq 2^{k-2} \quad \textit{and} \quad |\mathscr{L}_k| \leq 2^{k-2}. \tag{10}$$

Also, for all integers $k \geq 2$, the following holds:

$$\mathscr{U}_k \geq \binom{k-2}{\lceil \frac{k-2}{2} \rceil} \quad \textit{and} \quad |\mathscr{L}_k| \geq \binom{k-2}{\lceil \frac{k-2}{2} \rceil}. \tag{11}$$

Theorem 3 implies that $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ grow exponentially in k. Due to Stirling's formula, these sequences are between $\frac{2^{k-2}}{k-2}$ and 2^{k-2}.

We conjecture that (11) are equalities. However, we have formal proofs only for $k \in \{1,2,3,4\}$ and we believe that it is possible to obtain at least a computer-assisted proof for $k = 5$. We don't have the proof yet.

2. Essential and influential edges

Except for Proposition 1 below, the results in this section were proved in [4]. Proposition 1 is trivial, but so important that it must be listed.

Proposition 1. *For every $i \neq j$, every $\alpha, \beta \in \{a, b\}$, and every random variable φ,*

$$\sigma_i^\alpha \circ \sigma_i^\beta = \sigma_i^\alpha; \tag{12}$$

$$\sigma_i^\alpha \circ \sigma_j^\beta = \sigma_j^\beta \circ \sigma_i^\alpha; \tag{13}$$

$$(\partial_i \varphi) \circ \sigma_i^\alpha = \partial_i \varphi; \tag{14}$$

$$\partial_i \partial_i \varphi = 0; \tag{15}$$

$$\partial_i \partial_j \varphi = \partial_j \partial_i \varphi; \tag{16}$$

$$\varphi \cdot 1_{\omega_i = \alpha} = \varphi \circ \sigma_i^\alpha \cdot 1_{\omega_i = \alpha}. \tag{17}$$

For a fixed edge $j \in W_n$, define the events A_j and $\hat{A}_j$ with

$$A_j = \left\{\omega \in \Omega_n : \partial_j f(\omega) \neq 0\right\}; \tag{18}$$

$$\hat{A}_j = \left\{\omega \in \Omega_n : \partial_j f(\omega) = b-a\right\}. \tag{19}$$

The edge j is called *influential* if the event A_j occurred, and *very influential* if $\hat{A}_j$ occurred. The edge j is called *essential* if each geodesics passes through j, and *semi-essential* if at least one geodesic passes through j. We will denote by E_j the event that the edge j is essential and by $\hat{E}_j$ the event that the edge j is semi-essential.

Proposition 2. *The events E_j, A_j, $\hat{E}_j$, and $\hat{A}_j$ satisfy*

$$A_j = (\sigma_j^a)^{-1}(E_j); \tag{20}$$

$$\hat{A}_j = (\sigma_j^b)^{-1}(\hat{E}_j). \tag{21}$$

Proposition 3. *The following two propositions hold for every $\omega \in \Omega$.*

(a) *If $\sigma_j^a(\omega) \in E_j^C$, then $f(\sigma_j^b(\omega)) = f(\sigma_j^a(\omega))$;*
(b) *If $\sigma_j^b(\omega) \in \hat{E}_j$, then $f(\sigma_j^b(\omega)) = f(\sigma_j^a(\omega)) + (b-a)$.*

The sets $\{\omega_j = a\}$ and $\{\omega_j = b\}$ are the ranges of the transformations σ_j^a and σ_j^b, i.e.

$$\sigma_j^a(\Omega) = \{\omega_j = a\} \quad \text{and} \quad \sigma_j^b(\Omega) = \{\omega_j = b\}. \tag{22}$$

Proposition 4. *For every $j \in W$, the events E_j, $\hat{E}_j$, A_j, and $\hat{A}_j$ satisfy*

$$\sigma_j^a(A_j) = \sigma_j^a(A_j \cap \{\omega_j = b\}) = A_j \cap \left\{\omega_j = a\right\} = E_j \cap \left\{\omega_j = a\right\}; \tag{23}$$

$$\sigma_j^b(\hat{A}_j) = \sigma_j^b(\hat{A}_j \cap \{\omega_j = a\}) = \hat{A}_j \cap \left\{\omega_j = b\right\} = \hat{E}_j \cap \left\{\omega_j = b\right\}. \tag{24}$$

Proposition 5. *For every $j \in W$, the events E_j, $\hat{E}_j$, A_j, and $\hat{A}_j$ satisfy*

$$\begin{aligned} &E_j \subseteq \hat{E}_j, \quad \hat{A}_j \subseteq A_j, \\ &E_j \subseteq A_j, \quad \hat{A}_j \subseteq \hat{E}_j. \end{aligned} \tag{25}$$

Proposition 6. *Assume that $\omega \in E_j$. A path γ is a geodesic on ω if and only if it is a geodesic on $\sigma_j^a(\omega)$.*

If $\overrightarrow{\alpha} \in \{a,b\}^m$ and $\overrightarrow{v} \in W^m$, define $\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}} : \Omega \to \Omega$ as

$$\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}} = \sigma_{v_1}^{\alpha_1} \circ \cdots \circ \sigma_{v_m}^{\alpha_m}, \tag{26}$$

where $\alpha_1, \ldots, \alpha_m$ are the components of $\overrightarrow{\alpha}$ and $v_1, \ldots, v_m$ are the components of $\overrightarrow{v}$.

3. Disjoint lanes

Our goal is to construct special, extreme environments in which the derivatives attain very large positive values and very small negative values. These environments will be used to establish the bounds in (11) from Theorem 3.

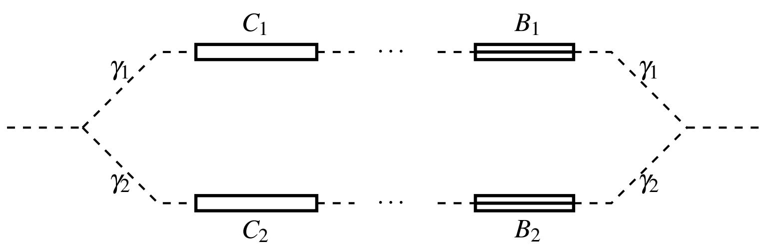
The proofs involve somewhat lengthy algebraic calculations. Such calculations are omitted here and are presented in the Appendix.

The results of this section apply to both models: the first-passage percolation between source and sink, and the first-passage percolation time on torus model.

Fix non-negative integers m_1, m_2, β_1, and β_2 for which $m_1 + m_2 \geq 2$. We investigate the environments in which the source and the sink are connected by two paths P_1 and P_2 that have two straight disjoint segments γ_1 and γ_2 that are sufficiently far apart. The set S is assumed to have $m = m_1 + m_2$ edges. Let $\overrightarrow{v} = (v_1, v_2, \dots, v_m)$ be the vector that contains all edges of S. The section γ_1 contains m_1 of the edges from S and the section γ_2 contains m_2 of the edges from S. We will denote by C_1 the set of m_1 edges from S that belong to γ_1. Similarly, C_2 is the set of m_2 edges from S that belong to γ_2. On γ_1 we fix a set B_1 of β_1 edges. On γ_2 we fix a set B_2 of β_2 edges. We will consider the environment ω that assigns the value b to all the edges from B_1 and B_2 and all the edges outside of $P_1 \cup P_2$. The environment ω assigns the value a to every edge in $P_1 \cap P_2 \setminus (B_1 \cup B_2)$.

We assume that n is sufficiently large to make it possible for the two paths P_1 and P_2 to have the following properties:

- The paths P_1 and P_2 are of equal lengths.
- The sections γ_1 and γ_2 are of equal lengths.
- The sections γ_1 and γ_2 are far away from each other.
- The sets B_1 and B_2 are far from each other and far from all of the edges from S.
- For each $\overrightarrow{x} \in \{a,b\}^{m_1+m_2}$, the only geodesics on $\sigma_{\overrightarrow{v}}^{\overrightarrow{\xi}}(\omega)$ belong to $\{P_1, P_2\}$. In other words, for any assignment of values to the edges of S, one or both of P_1, P_2 will be the geodesic, and there are no other geodesics.



Define $D(m_1, m_2; \beta_1, \beta_2)$ to be the environment derivative $\partial_S f(\omega)$ for the pair (S, ω) that was described above.

Theorem 4. *The number $D(m_1, m_2; \beta_1, \beta_2)$ is 0 if $\beta_1 - \beta_2 \geq m_2$ or $\beta_2 - \beta_1 \geq m_1$. If both of the inequalities $\beta_1 - \beta_2 \leq m_2 - 1$ and $\beta_2 - \beta_1 \leq m_1 - 1$ are satisfied, then*

$$D(m_1, m_2; \beta_1, \beta_2) \quad = \quad (b-a) \cdot (-1)^{m_1+m_2+\beta_1+\beta_2} \cdot \binom{m_1 + m_2 - 2}{m_1 + \beta_1 - \beta_2 - 1}. \tag{27}$$

Proof. The proof is in the Appendix. □

The next proposition implies the bounds (11) in Theorem 3.

Proposition 7. *For every $m \geq 2$, the following hold*

$$\mathscr{U}_m \geq \binom{m-2}{\lceil \frac{m-2}{2} \rceil} \quad \text{and} \quad \mathscr{L}_m \leq -\binom{m-2}{\lceil \frac{m-2}{2} \rceil}. \tag{28}$$

Proof. If m is odd, then the bound for $\mathscr{U}_m$ is attained when (27) is applied to $(m_1, m_2; \beta_1, \beta_2) = (\frac{m-1}{2}, \frac{m+1}{2}; 1, 0)$. The bound for $\mathscr{L}_m$ is attained for $(m_1, m_2; \beta_1, \beta_2) = (\frac{m+1}{2}, \frac{m-1}{2}; 0, 0)$.

If m is even, then the bound for $\mathscr{U}_m$ is attained for $(m_1, m_2; \beta_1, \beta_2) = (\frac{m}{2}, \frac{m}{2}, 0, 0)$, while the bound for $\mathscr{L}_m$ is attained for $(m_1, m_2; \beta_1, \beta_2) = (\frac{m}{2} - 1, \frac{m}{2} + 1; 1, 0)$. □

4. Almost sure bounds

In this section we prove the Theorem 3. We will first prove the inequalities (9). It suffices to prove the proposition below.

Proposition 8. *Assume φ is a random variable such that for every subset $T \subseteq W$ with k elements we have $\partial_T \varphi \in [L,U]$. Then, the following inequality holds for every subset $S \subseteq W$ with $k+1$ elements.*

$$\partial_S \varphi \in [L-U, U-L]. \tag{29}$$

Proof. Let s be an arbitrary element of S. Let $T = S \setminus \{s\}$.

$$\partial_S \varphi(\omega) \quad = \quad \partial_T \varphi(\sigma_s^b(\omega)) - \partial_T \varphi(\sigma_s^a(\omega)).$$

The result (29) immediately follows from the previous equality. □

Theorem 5. *Let $k \in W$ and let $S \subseteq W$ be a subset with at least two elements. The derivatives of the first-passage percolation time f satisfy the following inequalities for every $\omega \in \Omega$.*

$$\partial_k f(\omega) \quad \in \quad [0, b-a]; \tag{30}$$

$$\partial_S f(\omega) \quad \in \quad [-(b-a), b-a], \quad \text{if } |S| = 2; \tag{31}$$

$$|\partial_S f(\omega)| \quad \leq \quad 2^{|S|-2} \cdot (b-a). \tag{32}$$

Proof. The relation (30) is obvious because the function f must increase, and it can increase by at most $b-a$ if one edge changes its passage time from a to b. Let us first observe that for sets S with two elements, the relation (31) and the inequality (32) follow directly from (30) and (29). Observe that if φ is any function, and not just first passage percolation time, then (4) implies $|\partial_G \varphi(\omega)| \leq 2^{|G|} \|\varphi\|_\infty$ for every set G. Assume now that S has at least two elements k and l. Let $G = S \setminus \{k, l\}$.

$$\begin{aligned} |\partial_S f(\omega)| \quad &= \quad |\partial_G (\partial_k \partial_l f(\omega))| \leq 2^{|G|} \cdot \|\partial_k \partial_l f\|_\infty \\ &\leq \quad 2^{|G|} \cdot (b-a). \end{aligned}$$

The proof is complete once we observe that $|G| = |S| - 2$. □

5. Evaluation of $\mathscr{U}_3$, $\mathscr{L}_3$, $\mathscr{U}_4$, and $\mathscr{L}_4$

The upper and lower bounds can be improved when $|S| = 3$.

5.1. Upper bounds.

Theorem 6. *Let $S \subseteq W$ be a subset with three elements. The first passage percolation time f satisfies the following inequality for every $\omega \in \Omega$*

$$\partial_S f(\omega) \quad \leq \quad (b-a). \tag{33}$$

Proof. Let $S = \{k, l, m\}$. We will make our notation shorter and write $\sigma^{(\theta_1,\theta_2,\theta_3)}(\omega)$ instead of $\sigma_k^{\theta_1} \circ \sigma_l^{\theta_2} \circ \sigma_m^{\theta_3}(\omega)$ for $(\theta_1, \theta_2, \theta_3) \in \{a, b\}^3$. We will first prove that the inequality (33) is satisfied if

$$\sigma^{(a,a,a)}(\omega) \quad \in \quad E_k^C \cup E_l^C \cup E_m^C. \tag{34}$$

If we assume $\sigma^{(a,a,a)}(\omega) \in E_k^C$, then the Proposition 3 (a) implies that $f(\sigma^{(b,a,a)}(\omega)) = f(\sigma^{(a,a,a)}(\omega))$, i.e. $\partial_k f(\sigma^{(a,a,a)}(\omega)) = 0$. Then,

$$\begin{aligned} \partial_S f(\omega) \quad &= \quad \partial_k f(\sigma^{(a,b,b)}(\omega)) - \partial_k f(\sigma^{(a,a,b)}(\omega)) - \partial_k f(\sigma^{(a,b,a)}(\omega)) \\ &\leq \quad (b-a). \end{aligned}$$

We have proved that $\sigma^{(a,a,a)}(\omega) \in E_k^C$ implies $\partial_S f(\omega) \leq (b-a)$. In analogous ways we prove that the inequality (33) is implied if $\sigma^{(a,a,a)}(\omega)$ belongs to E_l^C or E_m^C.

We will now prove that (33) holds if

$$\sigma^{(b,a,b)}(\omega) \notin \hat{E}_k^C \cap E_l \cap \hat{E}_m^C. \tag{35}$$

If $\sigma^{(b,a,b)}(\omega) \in E_l^C$, then $\partial_l f(\sigma^{(b,a,b)}(\omega)) = 0$, due to Proposition 3 (a). The derivative $\partial_S f(\omega)$ becomes

$$\begin{aligned} \partial_S f(\omega) &= -\partial_l f(\sigma^{(a,a,b)}(\omega)) - \partial_l f(\sigma^{(b,a,a)}(\omega)) + \partial_l f(\sigma^{(a,a,a)}(\omega)) \\ &\leq -0 - 0 + (b-a) = b-a. \end{aligned}$$

Assume now that $\sigma^{(b,a,b)}(\omega) \in \hat{E}_k^C$. Proposition 3 (b) implies $\partial_k f(\sigma^{(a,a,b)}(\omega)) = b-a$. Therefore,

$$\begin{aligned} \partial_S f(\omega) &= -(b-a) + \partial_k f(\sigma^{(a,b,b)}(\omega) - \partial_k f(\sigma^{(a,b,a)}(\omega)) + \partial_k f(\sigma^{(a,a,a)}(\omega)) \\ &\leq -(b-a) + (b-a) + 0 + (b-a) = (b-a). \end{aligned}$$

In analogous way we prove that $\partial_S f(\omega) \leq (b-a)$ holds if any of the following two inclusions is satisfied:

$$\sigma^{(a,b,b)}(\omega) \notin E_k \cap \hat{E}_l^C \cap \hat{E}_m^C \quad \text{or} \quad \sigma^{(b,b,a)}(\omega) \notin \hat{E}_k^C \cap \hat{E}_l^C \cap E_m.$$

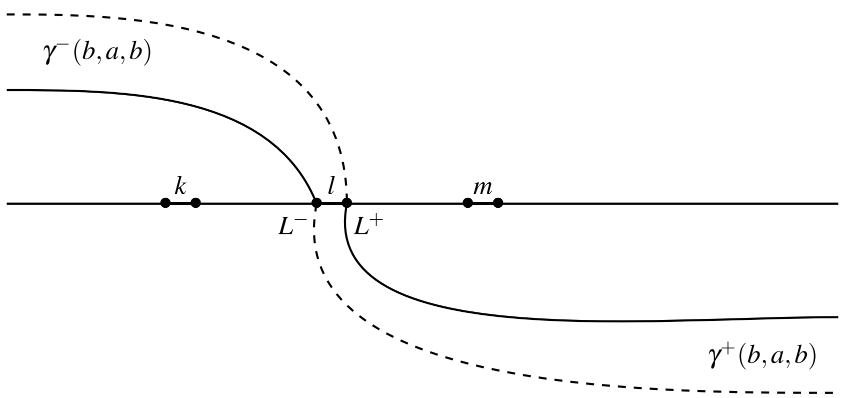
It remains to prove (33) if we assume that all of the following four conditions are satisfied

$$\sigma^{(a,b,b)}(\omega) \in E_k \cap \hat{E}_l^C \cap \hat{E}_m^C, \tag{36}$$

$$\sigma^{(b,a,b)}(\omega) \in \hat{E}_k^C \cap E_l \cap \hat{E}_m^C, \tag{37}$$

$$\sigma^{(b,b,a)}(\omega) \in \hat{E}_k^C \cap \hat{E}_l^C \cap E_m, \tag{38}$$

$$\sigma^{(a,a,a)}(\omega) \in E_k \cap E_l \cap E_m. \tag{39}$$



Let γ be a geodesic on $\sigma^{(a,a,a)}(\omega)$. All of the edges k, l, and m must belong to γ. Without loss of generality, assume that they appear in this order: k, l, m. Let $\gamma(b,a,b)$ be a geodesic on $\sigma^{(b,a,b)}(\omega)$. According to (37), the geodesic $\gamma(b,a,b)$ must pass through l and must not pass through either of k, m. Let $\gamma^-(b,a,b)$ and $\gamma^+(b,a,b)$ be the sections of $\gamma(b,a,b)$ before and after the edge l. The sections $\gamma^-(b,a,b)$ and $\gamma^+(b,a,b)$ are assumed not to contain the edge l. Let us denote by $T^-(b,a,b)$ and $T^+(b,a,b)$ the passage times over the sections $\gamma^-(b,a,b)$ and $\gamma^+(b,a,b)$.

Let L^- and L^+ be the left and right endpoints of the edge l with respect to γ. More precisely, on the geodesic γ, the endpoint L^- appears before the endpoint L^+. There are two possible orders of L^- and L^+ on the path $\gamma(b,a,b)$. The first possibility is that the endpoint L^- appears before L^+ on $\gamma^{(b,a,b)}$. This possibility is shown with solid line in the picture. The second possibility is that L^+ appears before L^-. The dashed line represents the case in which this occurs.

Let us denote by T^- the passage time on the environment $\sigma^{(a,a,a)}(\omega)$ over the segment of γ between the source and just before reaching the edge l. Denote by T^+ the passage time on $\sigma^{(a,a,a)}(\omega)$ over the segment after the edge l until the sink. The following equations must hold

$$f(\sigma^{(a,a,a)}(\omega)) = T^- + T^+ + a, \quad (40)$$

$$f(\sigma^{(b,a,b)}(\omega)) = T^-(b,a,b) + T^+(b,a,b) + a.. \quad (41)$$

In the case when L^- appears before L^+ on $\gamma(b,a,b)$, the following two inequalities are satisfied:

$$f(\sigma^{(a,a,b)}(\omega)) \leq T^- + T^+(b,a,b) + a, \quad (42)$$

$$f(\sigma^{(b,a,a)}(\omega)) \leq T^-(b,a,b) + T^+ + a. \quad (43)$$

If L^+ appears before L^-, then even stronger relations hold:

$$f(\sigma^{(a,a,b)}(\omega)) \leq T^- + T^+(b,a,b), \quad (44)$$

$$f(\sigma^{(b,a,a)}(\omega)) \leq T^-(b,a,b) + T^+. \quad (45)$$

Clearly, (44) and (45) imply (42) and (43) are satisfied. Hence, (42) and (43) hold always. The relations (40), (41), (42), and (43) imply

$$f(\sigma^{(a,a,b)}(\omega)) + f(\sigma^{(b,a,a)}(\omega)) - f(\sigma^{(b,a,b)}(\omega)) - f(\sigma^{(a,a,a)}(\omega)) \leq 0. \quad (46)$$

Therefore, the derivative $\partial_S f(\omega)$ can be bounded in the following way

$$\partial_S f(\omega) \leq \partial_k f(\sigma^{(a,b,b)}(\omega)) - \partial_k f(\sigma^{(a,b,a)}(\omega)). \quad (47)$$

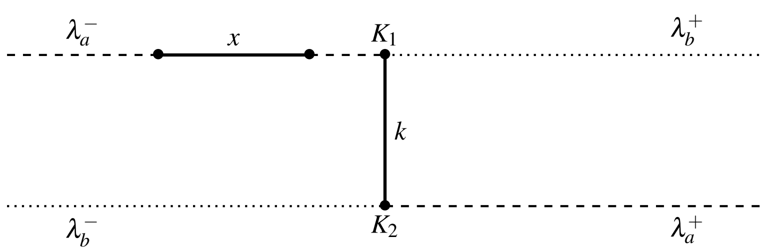
Since the derivatives of the first order belong to $[0, b-a]$, the first term on the right-hand side in (47) is smaller than or equal to $(b-a)$ while $\partial_k f(\sigma^{(a,b,a)}(\omega))$ is greater than or equal to 0. Therefore, (47) implies (33). □

5.2. **Direction switching.** In the next proposition we will prove that the direction of the flow cannot switch if only one edge is flipped from a to b.

Proposition 9. *Assume that k and x are two edges and that K_1 and K_2 are the endpoints of k. Assume that ω is a fixed environment such that on $\sigma_x^a(\omega)$ there is a geodesic λ_a that contains k and on $\sigma_x^b(\omega)$ there is a geodesic λ_b that contains k. Then, the order of K_1 and K_2 on λ_a is the same as their order on λ_b.*

Proof. We will only consider the case $k \neq x$. The case $k = x$ is easier, and it will be discussed in the remark below the proof. Assume the contrary, that K_1 appears before K_2 on λ_a, but K_2 appears before K_1 on λ_b. Let λ_a^- be the section of λ_a before K_1. Let λ_a^+ be the section of λ_a after K_2. Define λ_b^- and λ_b^+ as the sections of λ_b that appear before K_2 and after K_1, respectively.

Without loss of generality, assume that $x \in \lambda_a^-$.



The value ω_k from $\{a,b\}$ that is assigned to the edge k is the same on $\sigma_x^a(\omega)$ and $\sigma_x^b(\omega)$. The path $\lambda_a = \lambda_a^- \cup \{k\} \cup \lambda_a^+$ is a geodesic on $\sigma_x^a(\omega)$, while $\lambda_a^- \cup \lambda_b^+$ is just a path from the source to the sink. Therefore, we must have

$$\omega_k + T(\lambda_a^+) \quad \leq \quad T(\lambda_b^+). \tag{48}$$

The path λ_b is a geodesic on $\sigma_x^b(\omega)$. The path $\lambda_b^- \cup \lambda_a^+$ does not have to be a geodesic, hence

$$\omega_k + T(\lambda_b^+) \quad \leq \quad T(\lambda_a^+). \tag{49}$$

If we add (48) and (49), we obtain $2\omega_k \leq 0$. This is a contradiction. □

Remark. In the case $k = x$, a slight modification to the proof consists of replacing ω_k by a in (48) and by b in (49). The conclusion $a + b \leq 0$ leads to contradiction in this case.

Definition 1. *Assume that k, l, and m are three different fixed edges. Edge k is called direction switching edge with respect to the pair of edges (l,m) on the environment ω if its two endpoints K_1 and K_2 satisfy the following condition: On the environment $\sigma_k^a \sigma_l^a \sigma_m^b(\omega)$ there is a geodesic such that K_1 is before K_2; while on the environment $\sigma_k^a \sigma_l^b \sigma_m^a(\omega)$ there is a geodesic such that K_2 is before K_1.*

Proposition 10. *Assume that $S = \{k,l,m\}$ and that k is a direction switching edge with respect to (l,m). Let K_1 and K_2 be the endpoints of k. Let λ be a geodesic on $\sigma_k^a \sigma_l^a \sigma_m^b(\omega)$ such that K_1 appears before K_2 on λ. Let μ be the geodesic on $\sigma_k^a \sigma_l^b \sigma_m^a(\omega)$ such that K_2 appears before K_1 on μ. Denote by λ^- the section of λ between the source and K_1 and by λ^+ the section of λ between K_2 and the sink. Define μ^- and μ^+ in analogous ways. Then, one of the following two mathematical propositions $P(\lambda^-,\mu^+)$ and $P(\lambda^+,\mu^-)$ is satisfied*

$$P(\lambda^-,\mu^+) \quad \equiv \quad \left\{(l \in \lambda^-) \quad \textit{and} \quad (m \in \mu^+)\right\}, \tag{50}$$

$$P(\lambda^+,\mu^-) \quad \equiv \quad \left\{(l \in \lambda^+) \quad \textit{and} \quad (m \in \mu^-)\right\}. \tag{51}$$

Proof. Let's first prove that we can't have $\{l,m\} \subseteq \lambda$ or $\{l,m\} \subseteq \mu$. Assume the contrary, that both of l and m belong to λ. The passage times over both λ and μ would not change if the environment changes from $\sigma_k^a \sigma_l^a \sigma_m^b(\omega)$ to $\sigma_k^a \sigma_l^b \sigma_m^a(\omega)$. Therefore, both μ and λ are geodesics on each of the two environments. Let us fix one of the two environments, say $\sigma_k^a \sigma_l^a \sigma_m^b(\omega)$. The path $\lambda^- \cup \mu^+$ is not a geodesic, hence

$$T(\lambda^-) + T(\mu^+) > T(\lambda^-) + T(k) + T(\lambda^+) = T(\lambda^-) + T(\lambda^+) + a.$$

In an analogous way we obtain

$$T(\mu^-) + T(\lambda^+) > T(\mu^-) + T(k) + T(\mu^+) = T(\mu^-) + T(\mu^+) + a.$$

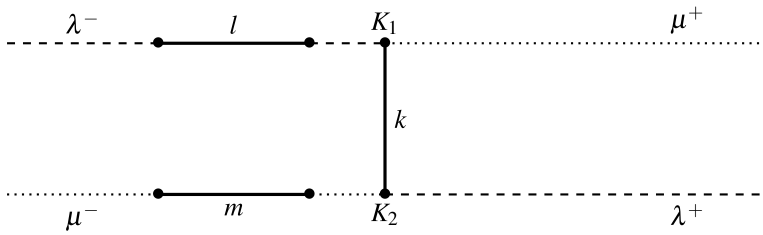
If we add the last two inequalities, we obtain $0 > 2a$, which is impossible.

Therefore, λ contains exactly one of the edges $\{l,m\}$ and μ contains the other one. We can't have $m \in \lambda$ and $l \in \mu$ because λ is a geodesic when the value a is assigned to l and value b is assigned to m; while μ is a geodesic when a is assigned to m and b is assigned to l. Therefore, $l \in \lambda$ and $m \in \mu$.

In order to prove that $P(\lambda^-,\mu^+)$ or $P(\lambda^+,\mu^-)$ is satisfied, we must prove that neither of the following two propositions $P(\lambda^-,\mu^-)$, $P(\lambda^+,\mu^+)$ can hold.

$$P(\lambda^-,\mu^-) \quad \equiv \quad \left\{(l \in \lambda^-) \quad \text{and} \quad (m \in \mu^-)\right\}, \tag{52}$$

$$P(\lambda^+,\mu^+) \quad \equiv \quad \left\{(l \in \lambda^+) \quad \text{and} \quad (m \in \mu^+)\right\}. \tag{53}$$



Assume that $P(\lambda^-,\mu^-)$ is satisfied. Then, since $\lambda^- \cup \mu^+$ is not a geodesic on the environment $\sigma_k^a\sigma_l^a\sigma_m^b(\omega)$, while λ is, we obtain

$$T(\lambda^- \setminus \{l\}) + a + T(\mu^+) \quad > \quad T(\lambda^- \setminus \{l\}) + 2a + T(\lambda^+). \tag{54}$$

Since $\mu^- \cup \lambda^+$ is not a geodesic on $\sigma_k^a\sigma_l^b\sigma_m^a(\omega)$, while μ is, we derive

$$T(\mu^- \setminus \{m\}) + a + T(\lambda^+) \quad > \quad T(\mu^- \setminus \{l\}) + 2a + T(\mu^+). \tag{55}$$

Adding the inequalities (54) and (55) implies $0 > 2a$, which is a contradiction. In an analogous way we treat the case in which $P(\lambda^+,\mu^+)$ holds.

Thus, we have proved that one of the mathematical propositions $P(\lambda^-,\mu^-)$, $P(\lambda^+,\mu^+)$ must be satisfied. $\square$

Proposition 11. *Assume that $S = \{k,l,m\}$ and that k is a direction switching edge with respect to (l,m). Then, the following two inequalities must hold*

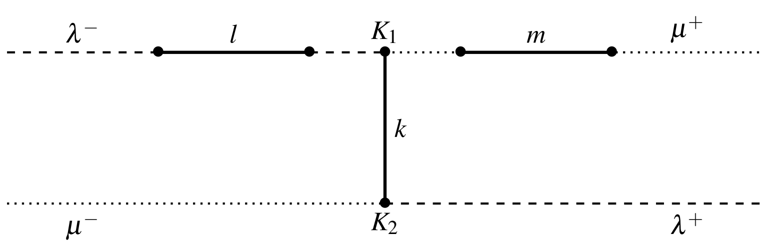
$$b \quad \geq \quad 3a, \tag{56}$$

$$\partial_S f(\omega) \quad \geq \quad 3a - b. \tag{57}$$

Proof. We will write $\sigma^{\vec{\xi}}(\omega)$ instead of $\sigma^{\vec{\xi}}_{(k,l,m)}(\omega)$. The following two inequalities are obvious because they follow from $\partial_z f(\omega) \geq 0$ for every edge z.

$$f(\sigma^{(b,b,b)}(\omega)) \geq f(\sigma^{(a,b,b)}(\omega)) \quad \text{and} \quad f(\sigma^{(a,a,b)}(\omega)) \geq f(\sigma^{(a,a,a)}(\omega)). \tag{58}$$

Let us define λ, μ, $\lambda^\pm$, and $\mu^\pm$ as in Proposition 10. The same proposition allows us to assume that $l \in \lambda^-$ and $m \in \mu^+$.



Since λ is a geodesic on $\sigma^{(a,a,b)}(\omega)$, we have

$$f(\sigma^{(a,a,b)}(\omega)) \quad = \quad T(\lambda^- \setminus \{l\}) + T(\lambda^+) + 2a. \tag{59}$$

In a similar way we obtain

$$f(\sigma^{(a,b,a)}(\omega)) \quad = \quad T(\mu^-) + T(\mu^+ \setminus \{m\}) + 2a. \tag{60}$$

The passage times over $\lambda^- \cup \mu^+$ on the environments $\sigma^{(b,a,b)}(\omega)$ and $\sigma^{(b,b,a)}(\omega)$ are the same because the path $\lambda^- \cup \mu^+$ contains both l and m. We will use T_1 to denote these two passage times, i.e. $T_1 = T(\lambda^- \cup \mu^+, \sigma^{(b,a,b)}(\omega))$.

The passage times over $\mu^- \cup \lambda^+$ are also the same on $\sigma^{(b,a,b)}(\omega)$ and $\sigma^{(b,b,a)}(\omega)$ because $\mu^- \cup \lambda^+$ contains neither l nor m. We will use T_2 to denote these passage times.

The shortest passage times on $\sigma^{(b,a,b)}(\omega)$ and $\sigma^{(b,b,a)}(\omega)$ are bounded by the number $\min\{T_1, T_2\}$. The numbers T_1 and T_2 satisfy

$$\begin{aligned} T_1 &= T(\lambda^- \setminus \{l\}) + T(\mu^+ \setminus \{m\}) + a + b, && (61)\\ T_2 &= T(\mu^-) + T(\lambda^+). && (62) \end{aligned}$$

We now use $f(\sigma^{(b,a,b)}(\omega)) \leq T_{\min}$, $f(\sigma^{(b,b,a)}(\omega)) \leq T_{\min}$, (58), (59), and (60) to conclude

$$\begin{aligned} \partial_S f(\omega) &\geq 4a + T(\lambda^- \setminus \{l\}) + T(\lambda^+) + T(\mu^-) + T(\mu^+ \setminus \{m\}) - 2\min\{T_1, T_2\} \\ &= 3a - b + T_1 + T_2 - 2\min\{T_1, T_2\}. \end{aligned}$$

It remains to observe that $T_1 + T_2 \geq 2\min\{T_1, T_2\}$, hence (57) is established.

In order to prove (56) we use that λ is a geodesic on $\sigma^{(a,a,b)}(\omega)$. The path $\mu^- \cup \lambda^+$ cannot have a shorter passage time than λ. Hence, the passage time over $\lambda^- \cup \{k\}$ must be smaller than or equal to the passage time over μ^-, i.e.

$$T(\mu^-) \geq T(\lambda^- \setminus \{l\}) + 2a. \tag{63}$$

Let us now analyze the environment $\sigma^{(a,b,a)}(\omega)$. On this environment, the path μ is a geodesic, while $\lambda^- \cup \mu^+$ does not have to be a geodesic. Hence, the section $\mu^- \cup \{k\}$ has shorter passage time than λ^-, when the value b is assigned to l and the value a to k. Therefore,

$$T(\mu^-) + a \leq T(\lambda^- \setminus \{l\}) + b. \tag{64}$$

The inequalities (63) and (64) imply (56). □

5.3. **Lower bounds.**

Theorem 7. *Let $S \subseteq W$ be a subset with three elements. The first passage percolation time f satisfies the following inequality for every $\omega \in \Omega$*

$$\partial_S f(\omega) \geq -(b-a). \tag{65}$$

Proof. Let $S = \{k, l, m\}$. Due to Proposition 11 and $3a - b > -(b-a)$, we may assume that none of the edges is direction-switching with respect to the other two.

We will first prove the following implication

$$\sigma^{(a,a,b)}(\omega) \notin E_k \cap E_l \cap \hat{E}_m^C \implies \partial_S f(\omega) \geq -(b-a) \tag{66}$$

The result (66) will follow from the following two

$$\begin{aligned} \sigma^{(a,a,b)}(\omega) \in E_k^C &\implies \partial_S f(\omega) \geq -(b-a), && (67)\\ \sigma^{(a,a,b)}(\omega) \in \hat{E}_m &\implies \partial_S f(\omega) \geq -(b-a). && (68) \end{aligned}$$

The first step in proving (67) is to express the derivative $\partial_S f(\omega)$ as

$$\begin{aligned} \partial_S f(\omega) &= \Big(f(\sigma^{(b,b,b)}(\omega)) - f(\sigma^{(b,b,a)}(\omega))\Big) && (69)\\ &\quad + \Big(f(\sigma^{(b,a,a)}(\omega)) - f(\sigma^{(a,a,a)}(\omega))\Big) && (70)\\ &\quad - \Big(f(\sigma^{(b,a,b)}(\omega)) - f(\sigma^{(a,a,b)}(\omega))\Big) && (71)\\ &\quad - \Big(f(\sigma^{(a,b,b)}(\omega)) - f(\sigma^{(a,b,a)}(\omega))\Big). && (72) \end{aligned}$$

Assume that $\sigma^{(a,a,b)}(\omega) \in E_k^C$. The Proposition 3 (a) implies that the term (71) is equal to 0. The terms (69) and (70) are non-negative, and the negative term (72) is bounded below by $-(b-a)$, which proves (67). In order to prove (68), we start by expressing $\partial_S f(\omega)$ as

$$\begin{aligned} \partial_S f(\omega) &= \Big(f(\sigma^{(b,b,b)}(\omega)) - f(\sigma^{(b,b,a)}(\omega))\Big) && (73)\\ &\quad + \Big(f(\sigma^{(a,a,b)}(\omega)) - f(\sigma^{(a,a,a)}(\omega))\Big) && (74)\\ &\quad - \Big(f(\sigma^{(b,a,b)}(\omega)) - f(\sigma^{(b,a,a)}(\omega))\Big) && (75)\\ &\quad - \Big(f(\sigma^{(a,b,b)}(\omega)) - f(\sigma^{(a,b,a)}(\omega))\Big). && (76) \end{aligned}$$

Assume that $\sigma^{(a,a,b)}(\omega) \in \hat{E}_m$. The Proposition 3 (b) implies that the term (74) is equal to $(b-a)$. The term (73) is non-negative. The terms (75) and (76) are negative but bounded below by $-(b-a)$. Therefore, $\partial_S f(\omega)$ is bounded below by $(b-a)-2(b-a) = -(b-a)$. This completes the proof of (68).

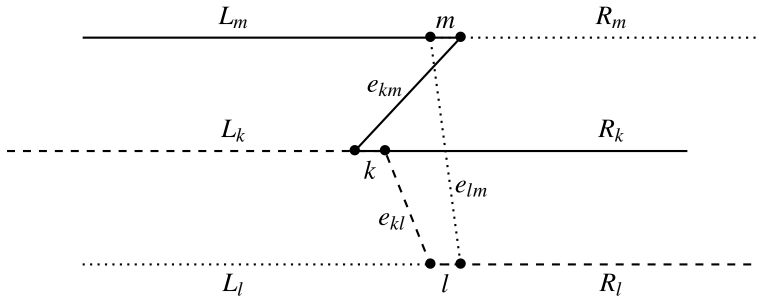
We proved (66) which states that the inequality $\partial_S f(\omega) \geq -(b-a)$ is satisfied unless $\sigma^{(a,a,b)}(\omega)$ is an element of $E_k \cap E_l \cap \hat{E}_m^C$. The analogous statements hold for $\sigma^{(a,b,a)}(\omega)$ and $\sigma^{(b,a,a)}(\omega)$. Hence, the required bound (65) is proved unless all of the following three inclusions are satisfied

$$\begin{aligned} \sigma^{(a,a,b)}(\omega) &\in E_k \cap E_l \cap \hat{E}_m^C, && (77)\\ \sigma^{(a,b,a)}(\omega) &\in E_k \cap \hat{E}_l^C \cap E_m, \quad \text{and} && (78)\\ \sigma^{(b,a,a)}(\omega) &\in \hat{E}_k^C \cap E_l \cap E_m. && (79) \end{aligned}$$

Hence, it suffices to prove $\partial_S f \geq -(b-a)$ under the conditions (77), (78), and (79). Let γ_{kl} be a geodesic on $\sigma^{(a,a,b)}(\omega)$ that does not contain the edge m. Such geodesic must exist because we assumed that $\sigma^{(a,a,b)}(\omega) \in \hat{E}_m^C$. The geodesic γ_{kl} must contain both edges k and l. We define the geodesics γ_{lm} and γ_{km} in analaogous ways. Once the paths γ_{kl}, γ_{lm}, and γ_{km} are fixed, we define the relation $\prec$ on $\{k,l,m\}$. We will write $k \prec l$ if on the path γ_{kl} the edge k appears before the edge l when moving from the source to the sink. There are two cases:

- Case 1: The relation $\prec$ does not have the minimum in $\{k,l,m\}$;
- Case 2: The relation $\prec$ has the minimum in $\{k,l,m\}$.

Case 1. This case is easier to consider. We will prove that $\partial_S f(\omega) > 0$ which is stronger than the required inequality. We may assume that $k \prec l$, $l \prec m$, and $m \prec k$.



The picture above represents the scenario in which the directions of the flow over the edges k, l, and m is the same on the environments $\sigma^{(a,a,b)}(\omega)$, $\sigma^{(a,b,a)}(\omega)$, and $\sigma^{(b,a,a)}(\omega)$. There are alternative situations that we will keep in mind throughout the proof, and these alternatives are actually easier to consider. We will denote by $A_m^{\leftrightarrow}$ the event when the direction of the flow over the edge m of γ_{lm} is the opposite from the direction of the flow of γ_{km}. We define the events $A_k^{\leftrightarrow}$ and $A_l^{\leftrightarrow}$ in analogous ways.

Let us denote by e_{kl} the total passage time between the edges k and l on the geodesic γ_{kl}. We define e_{km} and e_{lm} in analogous way. Let us denote by L_k the total passage time on the geodesic γ_{kl} before the edge k. Let R_l be the total passage time on the geodesic γ_{kl} after the edge l. The numbers L_m, L_l, R_k, and R_m are defined in similar ways. Let us emphasize that none of the previously defined passage times includes the edges k, m, and l.

Therefore, the quantities that we defined are the same on the environments $\sigma^{\overrightarrow{\theta}}(\omega)$ for all eight choices $\overrightarrow{\theta} \in \{a,b\}^3$. The following identities hold

$$\begin{aligned}
f(\sigma^{(a,a,b)}(\omega)) &= L_k + e_{kl} + R_l + 2a, && (80)\\
f(\sigma^{(a,b,a)}(\omega)) &= L_m + e_{km} + R_k + 2a, && (81)\\
f(\sigma^{(b,a,a)}(\omega)) &\geq f(\sigma^{(a,a,a)}(\omega)), && (82)\\
f(\sigma^{(b,b,b)}(\omega)) &\geq f(\sigma^{(b,b,a)}(\omega)), && (83)\\
f(\sigma^{(b,a,b)}(\omega)) &\leq L_l + a + R_l, && (84)\\
f(\sigma^{(a,b,b)}(\omega)) &\leq L_k + a + R_k. && (85)
\end{aligned}$$

The equalities (80) and (81) are due to the definitions of γ_{kl} and γ_{km}. The inequalities (82) and (83) are the consequences of monotonicity.

Let us prove (84). On the environment $\sigma^{(b,a,b)}(\omega)$, we can construct a path δ such that

$$T(\delta, \sigma^{(b,a,b)}(\omega)) = \begin{cases} L_l + a + R_l, & \text{if the event } A_l^{\leftrightarrow} \text{ did not occur,} \\ L_l + R_l, & \text{if the event } A_l^{\leftrightarrow} \text{ did occur.} \end{cases}$$

Let us identify the section of the path γ_{lm} before the edge l and call it δ_1. It has the passage time L_l. Let us consider the section of the path γ_{kl} after the edge l. This section will be called δ_2. Its passage time is R_l. If the event $A_l^{\leftrightarrow}$ did not occur, then $\delta = \delta_1 \cup \{l\} \cup \delta_2$ is the path between the source and the sink. If the event $A_l^{\leftrightarrow}$ did occur, then the section of γ_{kl} after the edge l starts at the entrance endpoint of γ_{lm} to the edge l. Therefore, $\delta = \delta_1 \cup \delta_2$ is a path from the source to the sink on $A_l^{\leftrightarrow}$. In the first case the passage time is $L_l + a + R_l$. In the second case the passage time is just $L_l + R_l$. In either case, the inequality (84) must hold.

The inequality (85) is proved in a similar way.

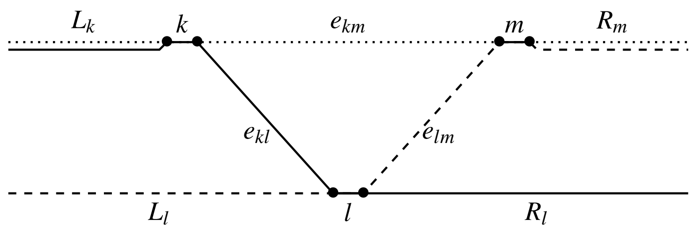
From (80)–(85) we obtain

$$\begin{aligned} \partial_S f(\omega) &\geq (L_k + e_{kl} + R_l + 2a) + (L_m + e_{km} + R_k + 2a) \\ &\quad -(L_l + a + R_l) - (L_k + a + R_k) \\ &= e_{kl} + L_m + e_{km} + 2a - L_l. \end{aligned} \tag{86}$$

Let us consider the geodesic γ_{lm} on the environment $\sigma^{(b,a,a)}(\omega)$. Let us consider $\hat{\hat{\xi}}$ that consists of the union of the left part of γ_{km} and the right part of γ_{lm}. If the event $A_m^{\leftrightarrow}$ did occur, then $\hat{\hat{\xi}}$ is already a path between the source and the sink that includes one endpoint of m, but not the entire edge m. Its passage time is $L_m + R_m$. If the event $A_m^{\leftrightarrow}$ did not occur, then the path $\xi = \hat{\hat{\xi}} \cup \{m\}$ is a path between the source and the sink whose passage time is $L_m + a + R_m$. The number $L_m + a + R_m$ is bigger than $L_m + R_m$. Hence, both on $A_m^{\leftrightarrow}$ and on its complement, we wound a path between source and the sink whose passage time is smaller than or equal to $L_m + R_m$. The path that we found is not a geodesic on $\sigma^{(b,a,a)}(\omega)$ because of (79). Hence, $L_m + a + R_m > L_l + e_{lm} + R_m + 2a$. The last inequality is equivalent to $L_m - L_l > e_{lm} + a$. The inequality (86) turns into $\partial_S f(\omega) \geq e_{kl} + e_{km} + e_{lm} + 3a > 3a > -(b-a)$. This finishes the proof in Case 1.

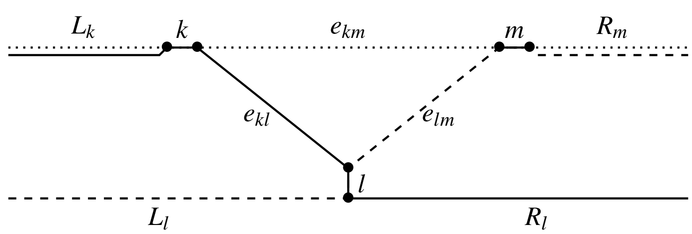
Remark. We proved that $\partial_S f > 3a$. If the number $3a$ were larger than $2(b-a)$, which is the maximal possible value for $\partial_S f$, then Case 1 would not be possible and Case 2 would be the only one worth considering.

Case 2. We may assume that k is the minimum, i.e. $k \prec l$ and $k \prec m$. Without loss of generality, we may assume that $l \prec m$.



On both γ_{kl} and γ_{km}, among the edges from S, the edge k appears the first. It is easy to prove that the direction of the flow over k is the same on γ_{kl} and γ_{km}. Similarly, the direction of the flow over m is the same on γ_{km} and γ_{lm}. However, we can't rule out that the direction of the flow over the edge l is the same on γ_{kl} and γ_{lm}. We will denote by $A_l^{\leftrightarrow}$ the event that the direction of the flow over the edge l is not the same on γ_{kl} and γ_{lm}. The picture above corresponds to the situation on which $A_l^{\leftrightarrow}$ did not occur. The picture below corresponds to the case in which the event $A_l^{\leftrightarrow}$ happens.

The argument that follows applies to both cases $(A_l^{\leftrightarrow})^C$ and $A_l^{\leftrightarrow}$.



The sections of the paths γ_{kl} and γ_{km} before the edge k must have equal passage times. Let us denote by L_k the common passage time of these sections. We may modify one of the paths γ_{kl} and γ_{km} in such a way that the sections before k actually coincide. In a similar way, the passage times after the edge m on the paths γ_{km} and γ_{lm} are equal. We will denote these passage times by R_m. We define L_l as the passage time over the path γ_{lm} before the edge l; R_l the passage time over γ_{kl} after l. We define e_{kl}, e_{lm}, and e_{km} as passage times over the open intervals (k,l), (l,m), and (k,m) on the paths γ_{kl}, γ_{lm}, and γ_{km}, respectively. Let us define the real number θ in such a way that the following holds

$$e_{km} = e_{kl}+e_{lm}+a+\theta. \tag{87}$$

Such θ is unique as it is a solution to a simple linear equation. We don't know whether θ is positive or negative. However, we will be able to prove that θ satisfies the inequality

$$\theta \leq b-a, \tag{88}$$

which is sufficient for our needs. On the environment $\sigma^{(a,b,a)}(\omega)$, the minimal passage time is over the path γ_{km}. This passage time is $L_k+e_{km}+R_m+2a$. On the event $(A_l^{\leftrightarrow})^C$, consider the path in which the segment (k,m) is replaced with $(k,l)\cup\{l\}\cup(l,m)$ (i.e., the section of γ_{kl} between k and l, the edge l, and the section of $\gamma_{l,m}$ between l and m). On the event $A_l^{\leftrightarrow}$, the passage time over the modified path is $L_k+e_{kl}+e_{lm}+R_m+2a$, while on $(A_l^{\leftrightarrow})^D$, the passage time is $L_k+e_{kl}+e_{lm}+R_m+2a+b$. The latter number is larger than the former, hence

$$L_k+e_{kl}+e_{lm}+R_m+2a+b > L_k+e_{km}+R_m+2a,$$

which, together with (87), implies (88).

Let us define the real numbers θ_L and θ_R with the following two identities

$$L_l = L_k+e_{kl}+a+\theta_L, \tag{89}$$

$$R_l = R_m+e_{lm}+a+\theta_R. \tag{90}$$

The numbers θ_L and θ_R must belong to the interval $(0,b-a]$. Let us prove that $\theta_L \in (0,b-a)$. We need to prove $\theta_L>0$ and $\theta_L\leq b-a$.

In order to prove $\theta_L>0$, we start by observing that on $\sigma^{(a,a,b)}(\omega)$, every geodesic must go through k. The path γ_{kl} is a geodesic. Let us take the section of this geodesic before the edge l and replace it with the corresponding section of γ_{lm}. On the event $(A_l^{\leftrightarrow})^C$ this replacement section does not contain l. On $A_l^{\leftrightarrow}$, the replacement does contain l. We obtain a path that is not a geodesic because of (77). On the event $(A_l^{\leftrightarrow})^C$, the change of passage time must satisfy

$$0 < L_l-L_k-a-e_{kl}=\theta_L. \tag{91}$$

On $A_l^{\leftrightarrow}$ a stronger inequality holds: $L_l+a>L_k+a+e_{kl}$. However, this stronger inequality implies (91).

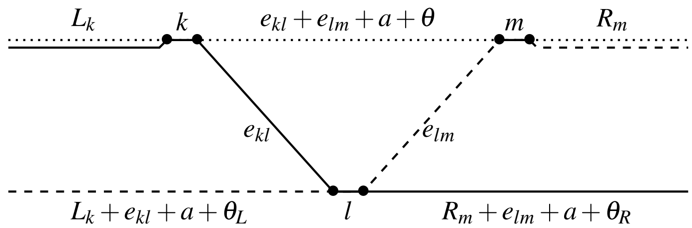
Let us now prove that $\theta_L\leq b-a$. Consider the environment $\sigma^{(b,a,a)}(\omega)$. The path γ_{lm} is a geodesic. Let us denote by γ_{kl}^- the section of γ_{kl} before the edge l. Let γ_{lm}^+ be the section of γ_{lm} after the edge l. On the event $(A_l^{\leftrightarrow})^C$, the path $\gamma_{kl}^-\cup\{l\}\cup\gamma_{lm}^+$ is a path from the source to the sink that has passage time $L_k+e_{kl}+e_{lm}+R_m+2a+b$. On the event $A_l^{\leftrightarrow}$, the union $\gamma_{kl}^-\cup\gamma_{lm}^+$ is a path from the source to the sink with the passage time $L_k+e_{kl}+e_{lm}+R_m+a+b$, which is smaller than $L_k+e_{kl}+e_{lm}+R_m+2a+b$. The passage time over the geodesic γ_{lm} is $L_l+e_{lm}+R_m+2a$. From

$$L_l+e_{lm}+R_m+2a \leq L_k+e_{kl}+e_{lm}+R_m+2a+b,$$

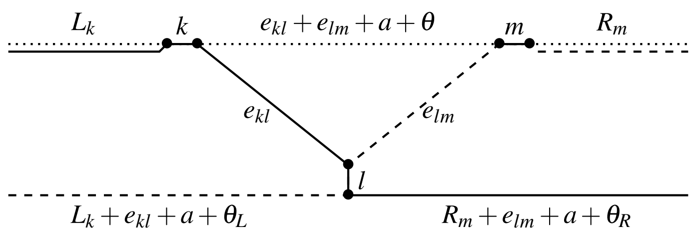
we obtain $L_l\leq L_k+e_{kl}+b$. The last inequality and (89) imply $\theta_L\leq(b-a)$.

In an analogous way we prove that $\theta_R\in(0,b-a]$.

We can now update our pictures. The next diagram represents the passage times on the event $(A_l^{\leftrightarrow})^C$.



The picture below represent the passage times on $A_l^{\leftrightarrow}$.



Since γ_{kl}, γ_{lm}, and γ_{km} are geodesics on $\sigma^{(a,a,b)}(\omega)$, $\sigma^{(b,a,a)}(\omega)$, and $\sigma^{(a,b,a)}(\omega)$, respectively, we obtain

$$f(\sigma^{(a,a,b)}(\omega)) = L_k+R_m+e_{kl}+e_{lm}+3a+\theta_R, \tag{92}$$
$$f(\sigma^{(a,b,a)}(\omega)) = L_k+R_m+e_{kl}+e_{lm}+3a+\theta, \tag{93}$$
$$f(\sigma^{(b,a,a)}(\omega)) = L_k+R_m+e_{kl}+e_{lm}+3a+\theta_L. \tag{94}$$

Due to monotonicity we have

$$f(\sigma^{(b,b,b)}(\omega)) \geq f(\sigma^{(a,b,b)}(\omega)). \tag{95}$$

Let us consider the environment $\sigma^{(b,a,b)}(\omega)$. We will construct a path ζ that has a low passage time and that will be a useful bound for $f(\sigma^{(b,a,b)}(\omega))$. Let γ_{lm}^- be the section of γ_{lm} before l. Let γ_{kl}^+ be the section of γ_{kl} after l. On the event $(A_l^{\leftrightarrow})^C$, we define $\zeta=\gamma_{lm}^-\cup\{l\}\cup\gamma_{kl}^+$. On the event $A_l^{\leftrightarrow}$, we define $\zeta=\gamma_{lm}^-\cup\gamma_{kl}^+$. In each of the cases, the passage time over ζ is smaller than or equal to $L_k+ e_{kl}+ \theta_L+ R_m+ e_{lm}+ \theta_R+ 3a$. The passage time over ζ is greater than or equal than the minimal passage time, hence

$$f(\sigma^{(b,a,b)}(\omega)) \leq L_k+R_m+e_{kl}+e_{lm}+3a+\theta_L+\theta_R. \tag{96}$$

The minimal passage time $f(\sigma^{(b,b,a)}(\omega))$ is smaller than or equal to the minimum of the passage times over the paths γ_{lm} and γ_{km}, hence

$$f(\sigma^{(b,b,a)}(\omega)) \leq L_k+R_m+e_{kl}+e_{lm}+2a+b+\min\{\theta_L,\theta\}. \tag{97}$$

Finally, let us consider the environment $\sigma^{(a,a,a)}(\omega)$. By considering the passage time over the path γ_{km}, we obtain

$$f(\sigma^{(a,a,a)}(\omega)) \leq L_k+R_m+e_{kl}+e_{lm}+3a+\theta. \tag{98}$$

Let us construct an alternative path γ' on the environment $\sigma^{(a,a,a)}(\omega)$. We replace the section (k,m) with the sections (k,l) and (l,m) of the paths γ_{kl} and γ_{lm}. On the event $(A_l^{\leftrightarrow})^C$ we must also add the edge l in order for γ' to connect source to the sink. The passage time over γ' is $L_k + R_m + e_{kl} + e_{lm} + 3a$ on $(A_l^{\leftrightarrow})^C$. On $A_l^{\leftrightarrow}$, the passage time over γ' is gives us $L_k + R_m + e_{kl} + e_{lm} + 2a$. Either way, we have

$$f(\sigma^{(a,a,a)}(\omega)) \leq L_k + R_m + e_{kl} + e_{lm} + 3a. \tag{99}$$

The inequalities (98) and (99) imply

$$f(\sigma^{(a,a,a)}(\omega)) \leq L_k + R_m + e_{kl} + e_{lm} + 3a + \min\{\theta, 0\}. \tag{100}$$

We now use (92)–(97) and (100) to find the lower bound on $\partial_S f(\omega)$. Observe that $R_k + R_m + e_{kl} + e_{lm}$ appears equally many times with sign $+$ as with sign $-$. We can ignore these terms. Hence,

$$\partial_S f(\omega) \geq a - b + \theta - \min\{\theta_L, \theta\} - \min\{\theta, 0\}. \tag{101}$$

Define $F(\theta, \theta_L) = \theta - \min\{\theta_L, \theta\} - \min\{\theta, 0\}$. It sufices to prove that $F(\theta, \theta_L) \geq 0$. There are two cases: $\theta \geq 0$ and $\theta < 0$. If $\theta \geq 0$, then $\min\{\theta, 0\} = 0$ and $F(\theta, \theta_L) = \theta - \min\{\theta_L, \theta\} \geq 0$. If $\theta < 0$, then from $\theta_L > 0$ we have $\min\{\theta, \theta_L\} = \theta$, and $F(\theta, \theta_L) = \theta - \theta - \theta = -\theta > 0$. This completes the proof of Case 2, which was the only remaining case that we needed to consider. □

Theorem 8. *The first four elements of the sequence $(\mathscr{U}_n)$ and the first three elements of the sequence $(\mathscr{L}_n)$ are*

$$(\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3, \mathscr{U}_4) = (1, 1, 1, 2); \tag{102}$$

$$(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4) = (0, -1, -1, -2). \tag{103}$$

Proof. The inequalities $\mathscr{U}_1 \leq 1$ and $\mathscr{L}_1 \geq 0$ follow from (30). The bound $\mathscr{U}_1 \geq 1$ can be proved by constructing an environment ω for which there is an edge k such that $\partial_k f(\omega) = b - a$. This is easy to do: Let ω^b be the environment that b to every edge. The equality $\partial_j f(\omega^b) = b - a$ holds for every edge j on the shortest path between the source and the sink. The bound $\mathscr{L}_1 \leq 0$ is equally easy to prove – the environment ω^a that assigns a to every edge satisfies $\partial_k f(\omega^a) = 0$ for every k.

The inequalities $\mathscr{U}_2 \leq 1$ and $\mathscr{L}_2 \geq -1$ follow from (31). Proposition 7 implies $\mathscr{L}_2 \leq -1$ and $\mathscr{U}_2 \geq 1$.

In addition, Proposition 7 also implies $\mathscr{U}_3 \geq 1$, $\mathscr{L}_3 \leq -1$, $\mathscr{U}_4 \geq 2$, $\mathscr{L}_4 \leq -2$. Theorem 6 implies $\mathscr{U}_3 \leq 1$ and Theorem 7 implies $\mathscr{L}_3 \geq -1$. Therefore, $\mathscr{U}_3 = 1$ and $\mathscr{L}_3 = -1$. The first inequality in (9) implies that $\mathscr{U}_4 \leq \mathscr{U}_3 - \mathscr{L}_3 = 2$. The second inequality in (9) implies that $\mathscr{L}_4 \geq \mathscr{L}_3 - \mathscr{U}_3 = -2$. □

Appendix A. Evaluations of derivatives in extreme environments

Proposition 12. *If A and B are two integers such that $0 \leq A \leq B$, then*

$$\sum_{k=A}^{B} (-1)^k \binom{B}{k} = \begin{cases} (-1)^A \cdot \binom{B-1}{A-1}, & A \geq 1, \\ 0, & A = 0. \end{cases} \tag{104}$$

Proof. In the case $A = 0$, the sum on the left-hand side becomes $(1-1)^B$, which is 0.

It remains to consider the case $A \geq 1$. Notice that for $B - 1 \geq k$ the following holds: $\binom{B}{k} = \binom{B-1}{k} + \binom{B-1}{k-1}$.

$$\begin{aligned} \sum_{k=A}^{B} (-1)^k \binom{B}{k} &= (-1)^B \binom{B}{B} + \sum_{k=A}^{B-1} (-1)^k \binom{B}{k} \\ &= (-1)^B + \sum_{k=A}^{B-1} (-1)^k \binom{B-1}{k} + \sum_{k=A}^{B-1} (-1)^k \binom{B-1}{k-1}. \end{aligned}$$

We substitute $k' = k-1$ in the second summation.

$$\begin{aligned}\sum_{k=A}^{B}(-1)^k\binom{B}{k} &= (-1)^B + \sum_{k=A}^{B-1}(-1)^k\binom{B-1}{k} - \sum_{k'=A-1}^{B-2}(-1)^{k'}\binom{B-1}{k'} \\ &= (-1)^B + (-1)^{B-1}\cdot\binom{B-1}{B-1} - (-1)^{A-1}\cdot\binom{B-1}{A-1} \\ &= (-1)^A\cdot\binom{B-1}{A-1}.\end{aligned}$$

This completes the proof of (104). □

Proposition 13. *If n and k are non-negative integers, and A an integer such that $n+A \in [0,k+n]$, the following holds*

$$\sum_{i=\max\{0,A\}}^{\min\{k,n+A\}}\binom{k}{i}\cdot\binom{n}{n+A-i} = \binom{n+k}{n+A}. \tag{105}$$

Proof. This is a famous cats-and-dogs identity. Assume that there are k cats and n dogs and that we want to count the number of ways to choose a committee consisting of $n+A$ animals. One obvious way to count committees is to ignore the differences between cats and dogs. The number becomes $\binom{n+k}{n+A}$. However, we can also do a counting by doing the case-work. If we denote by i the number of cats in the committee, then the number i must range from $\max\{0,A\}$ to $\min\{k,n+A\}$. The number of $(n+A)$-member committees with exactly i cats is $\binom{k}{i}\cdot\binom{n}{n+A-i}$. □

Proof of Theorem 4. Let L be the total number of edges on each of the paths P_1 and P_2. Assume, first, that $\beta_1-\beta_2 \geq m_2$. The path P_2 is a geodesic on every environment because the passage time over P_1 is larger than or equal to β_1, while the passage time over P_2 is smaller than or equal to $La+(m_2+\beta_2)\cdot(b-a)$. Hence, if i_2 of the edges from C_2 have the value b, then the shortest passage time is equal to $La+(i_2+\beta_2)(b-a)$. Hence,

$$\begin{aligned}\partial_S f(\omega) &= (b-a)\cdot\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}(-1)^{m_1-i_1+m_2-i_2}\cdot\binom{m_1}{i_1}\cdot\binom{m_2}{i_2}\cdot(i_2+\beta_2) \\ &= (b-a)\cdot\left(\sum_{i_1=0}^{m_1}(-1)^{m_1+i_1}\binom{m_1}{i_1}\right)\cdot\left(\sum_{i_2}^{m_2}(-1)^{m_2+i_2}\cdot\binom{m_2}{i_2}\cdot(i_2+\beta_2)\right).\end{aligned}$$

The first term of the product is equal to $(1-1)^{m_1}$, hence $\partial_S f(\omega)=0$. The case $\beta_2-\beta_1 \geq m_1$ is analogous.

We now treat the case in which both $\beta_1-\beta_2 \leq m_2-1$ and $\beta_2-\beta_1 \leq m_1-1$. Assume that i_1 of the edges from C_1 and i_2 edges from C_2 are assigned the value b. The passage time over the path P_1 is $La+(b-a)(i_1+\beta_1)$. The passage time over P_2 is $La+(b-a)(i_2+\beta_2)$. Therefore, the derivative satisfies

$$\begin{aligned}\partial_S f(\omega) &= (b-a)\cdot(-1)^{m_1+m_2}\times \\ &\quad\times\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}(-1)^{i_1+i_2}\cdot\binom{m_1}{i_1}\cdot\binom{m_2}{i_2}\cdot\min\{i_1+\beta_1,i_2+\beta_2\}.\end{aligned}$$

Let us express the derivative as $\partial_S f(\omega)=(b-a)\cdot(-1)^{m_1+m_2}\cdot(S_1+S_2)$, where the summation S_1 is carried over the pairs (i_1,i_2) for which $i_1+\beta_1 \leq i_2+\beta_2$, and the summation S_2 is over the pairs (i_1,i_2) for which $i_1+\beta_1 \geq i_2+\beta_2+1$.

If $i_1+\beta_1 \le i_2+\beta_2$, then $i_1 \le i_2+\beta_2-\beta_1 \le m_2+\beta_2-\beta_1$. Therefore, the number S_1 satisfies

$$\begin{aligned} S_1 &= \sum_{i_1=0}^{\mu_1}(-1)^{i_1}\cdot(i_1+\beta_1)\cdot\binom{m_1}{i_1}\cdot\sum_{i_2=\tau_2(i_1)}^{m_2}(-1)^{i_2}\cdot\binom{m_2}{i_2}, \quad \text{where} \qquad (106)\\ \mu_1 &= \min\{m_1, m_2+\beta_2-\beta_1\} \quad \text{and}\\ \tau_2(i_1) &= \max\{0, i_1+\beta_1-\beta_2\}. \end{aligned}$$

In a similar way we find that S_2 satisfies

$$\begin{aligned} S_2 &= \sum_{i_2=0}^{\mu_2}(-1)^{i_2}\cdot(i_2+\beta_2)\cdot\binom{m_2}{i_2}\cdot\sum_{i_1=\tau_1(i_2)}^{m_1}(-1)^{i_1}\cdot\binom{m_1}{i_1}, \quad \text{where} \qquad (107)\\ \mu_2 &= \min\{m_2, m_1+\beta_1-\beta_2-1\} \quad \text{and}\\ \tau_1(i_2) &= \max\{0, i_2+\beta_2-\beta_1+1\}. \end{aligned}$$

The summation S_1 will now be broken in two: The first, S_{11} contains the terms that correspond to i_1 in the range $[0,\beta_2-\beta_1]$. If $\beta_2-\beta_1<0$, then there is no summation S_{11}. The summation S_{12} contains the terms i_1 in the range $[\beta_2-\beta_1+1,\mu_1]$. We will prove that S_{11} is 0. The summation S_{12} will have simple lower bound $\tau_2(i_1)$ that is always equal to $i_1+\beta_1-\beta_2$. The number S_{11} is obviously 0 if $\beta_2<\beta_1$. If $\beta_2\ge\beta_1$ and $i_1\le\beta_2-\beta_1$, then $i_1+\beta_1-\beta_2\le 0$. This implies that $\tau_2(i_1)=0$. The number S_{11} becomes

$$S_{11} = \sum_{i_1=0}^{\beta_2-\beta_1}(-1)^{i_1}\cdot(i_1+\beta_1)\cdot\binom{m_1}{i_1}\cdot\sum_{i_2=0}^{m_2}(-1)^{i_2}\cdot\binom{m_2}{i_2}$$

The inner summation $\sum_{i_2=0}^{m_2}(-1)^{i_2}\cdot\binom{m_2}{i_2}$ is equal to 0 because it is equal to $(1-1)^{m_2}$. Therefore, $S_{11}=0$ and $S_1=S_{12}$, i.e.

$$S_1 = \sum_{i_1=\max\{0,\beta_2-\beta_1+1\}}^{\min\{m_1,m_2+\beta_2-\beta_1\}}(-1)^{i_1}\cdot(i_1+\beta_1)\cdot\binom{m_1}{i_1}\cdot\sum_{i_2=\tau_2(i_1)}^{m_2}(-1)^{i_2}\cdot\binom{m_2}{i_2}. \qquad (108)$$

The equality (104) transforms (108) into

$$\begin{aligned} S_1 &= \sum_{i_1=\max\{0,\beta_2-\beta_1+1\}}^{\min\{m_1,m_2+\beta_2-\beta_1\}}(-1)^{i_1}\cdot(i_1+\beta_1)\cdot\binom{m_1}{i_1}\cdot(-1)^{\tau_2(i_1)}\cdot\binom{m_2-1}{\tau_2(i_1)-1}\\ &= (-1)^{\beta_1-\beta_2}\cdot\sum_{i_1=\max\{0,\beta_2-\beta_1+1\}}^{\min\{m_1,m_2+\beta_2-\beta_1\}}(i_1+\beta_1)\cdot\binom{m_1}{i_1}\cdot\binom{m_2-1}{i_1+\beta_1-\beta_2-1}. \qquad (109) \end{aligned}$$

We treat the summation S_2 in a similar way. As before, S_{21} gathers the terms for which i_2 is in the range $[0,\beta_1-\beta_2-1]$. Of course, if $\beta_1-\beta_2-1<0$, then there is no summation S_{21}. The summation S_{22} contains the terms for which i_2 is in the range $[\beta_1-\beta_2,\mu_2]$. It is straightforward to prove that S_{21} is 0. Therefore,

$$S_2 = (-1)^{\beta_2-\beta_1+1}\cdot\sum_{i_2=\max\{0,\beta_1-\beta_2\}}^{\min\{m_2,m_1+\beta_1-\beta_2-1\}}(i_2+\beta_2)\cdot\binom{m_2}{i_2}\cdot\binom{m_1-1}{i_2+\beta_2-\beta_1}. \qquad (110)$$

The summations S_1 and S_2 have the opposite signs. The summation S_1 has the term $(i_1+\beta_1)$ that makes it possible to split the summation into two simpler ones T_{11} and T_{12}. In a similar way we split S_2 into T_{21} and T_{22}.

$$\begin{aligned}
S_1 &= (-1)^{\beta_1+\beta_2}\cdot(T_{11}+T_{12}), \quad S_2=-(-1)^{\beta_1+\beta_2}\cdot(T_{21}+T_{22}), \quad \text{where} && (111)\\
T_{11} &= \sum_{i_1=\max\{0,\beta_2-\beta_1+1\}}^{\min\{m_1,m_2+\beta_2-\beta_1\}} i_1\cdot\binom{m_1}{i_1}\cdot\binom{m_2-1}{i_1+\beta_1-\beta_2-1}, && (112)\\
T_{12} &= \beta_1\cdot\sum_{i_1=\max\{0,\beta_2-\beta_1+1\}}^{\min\{m_1,m_2+\beta_2-\beta_1\}} \binom{m_1}{i_1}\cdot\binom{m_2-1}{i_1+\beta_1-\beta_2-1}, && (113)\\
T_{21} &= \sum_{i_2=\max\{0,\beta_1-\beta_2\}}^{\min\{m_2,m_1+\beta_1-\beta_2-1\}} i_2\cdot\binom{m_2}{i_2}\cdot\binom{m_1-1}{i_2+\beta_2-\beta_1}, && (114)\\
T_{22} &= \beta_2\cdot\sum_{i_2=\max\{0,\beta_1-\beta_2\}}^{\min\{m_2,m_1+\beta_1-\beta_2-1\}} \binom{m_2}{i_2}\cdot\binom{m_1-1}{i_2+\beta_2-\beta_1}. && (115)
\end{aligned}$$

The summations T_{11} and T_{21} can be simplified by first observing that the summations cannot ever have a term corresponding to the index 0. Then for $i_1, i_2 \neq 0$, we use $i_1\cdot\binom{m_1}{i_1} = m_1\cdot\binom{m_1-1}{i_1-1}$ and $i_2\cdot\binom{m_2}{i_2} = m_2\cdot\binom{m_2-1}{i_2-1}$. After using these identities, we can shift the indices i_1 and i_2 by 1 and obtain

$$\begin{aligned}
T_{11} &= m_1\cdot\sum_{i_1=\max\{0,\beta_2-\beta_1\}}^{\min\{m_1-1,m_2+\beta_2-\beta_1-1\}} \binom{m_1-1}{i_1}\cdot\binom{m_2-1}{i_1+\beta_1-\beta_2}, && (116)\\
T_{21} &= m_2\cdot\sum_{i_2=\max\{0,\beta_1-\beta_2-1\}}^{\min\{m_2-1,m_1+\beta_1-\beta_2-2\}} \binom{m_2-1}{i_2}\cdot\binom{m_1-1}{i_2+1+\beta_2-\beta_1}. && (117)
\end{aligned}$$

In each of (116), (113), (117), and (115), we apply $\binom{n}{k} = \binom{n}{n-k}$ to the second binomial coefficient. The sums turn into

$$\begin{aligned}
T_{11} &= m_1\cdot\sum_{i_1=\max\{0,\beta_2-\beta_1\}}^{\min\{m_1-1,m_2+\beta_2-\beta_1-1\}} \binom{m_1-1}{i_1}\cdot\binom{m_2-1}{m_2-1-i_1-\beta_1+\beta_2}, && (118)\\
T_{12} &= \beta_1\cdot\sum_{i_1=\max\{0,\beta_2-\beta_1+1\}}^{\min\{m_1,m_2+\beta_2-\beta_1\}} \binom{m_1}{i_1}\cdot\binom{m_2-1}{m_2-i_1-\beta_1+\beta_2}, && (119)\\
T_{21} &= m_2\cdot\sum_{i_2=\max\{0,\beta_1-\beta_2-1\}}^{\min\{m_2-1,m_1+\beta_1-\beta_2-2\}} \binom{m_2-1}{i_2}\times\\
&\quad\times\binom{m_1-1}{m_1-1-i_2-1-\beta_2+\beta_1}, && (120)\\
T_{22} &= \beta_2\cdot\sum_{i_2=\max\{0,\beta_1-\beta_2\}}^{\min\{m_2,m_1+\beta_1-\beta_2-1\}} \binom{m_2}{i_2}\cdot\binom{m_1-1}{m_1-1-i_2-\beta_2+\beta_1}. && (121)
\end{aligned}$$

We apply (105) to each of (118), (119), (120), and (121).

$$T_{11} = m_1 \cdot \binom{m_1+m_2-2}{m_2-1+\beta_2-\beta_1}, \tag{122}$$

$$T_{12} = \beta_1 \cdot \binom{m_1+m_2-1}{m_2+\beta_2-\beta_1}, \tag{123}$$

$$T_{21} = m_2 \cdot \binom{m_1+m_2-2}{m_1+\beta_1-\beta_2-2} = m_2 \cdot \binom{m_1+m_2-2}{m_2+\beta_2-\beta_1}, \tag{124}$$

$$T_{22} = \beta_2 \cdot \binom{m_1-1+m_2}{m_1-1+\beta_1-\beta_2} = \beta_2 \cdot \binom{m_1+m_2-1}{m_2+\beta_2-\beta_1}. \tag{125}$$

From (122) and (124) we obtain

$$\begin{aligned} T_{11}-T_{21} &= \frac{(m_1+m_2-2)!\cdot((m_1+m_2)(\beta_2-\beta_1)+m_2)}{(m_2+\beta_2-\beta_1)!\cdot(m_1+\beta_1-\beta_2)!} \\ &= \binom{m_1+m_2-1}{m_2+\beta_2-\beta_1}\cdot\frac{((m_1+m_2)(\beta_2-\beta_1)+m_2)}{m_1+m_2-1}. \end{aligned} \tag{126}$$

We now use $S=(b-a)\cdot(-1)^{m_1+m_2}\cdot(S_1+S_2)$, (111), (126), (123), and (125) to obtain

$$\begin{aligned} \frac{(-1)^{m_1+m_2}}{b-a}S &= (-1)^{\beta_1+\beta_2}\cdot(T_{11}+T_{12}-T_{21}-T_{22}) \\ &= (-1)^{\beta_1+\beta_2}\cdot\binom{m_1+m_2-1}{m_2+\beta_2-\beta_1}\times \\ &\quad\times\left(\frac{((m_1+m_2)(\beta_2-\beta_1)+m_2)}{m_1+m_2-1}+\beta_1-\beta_2\right) \\ &= (-1)^{\beta_1+\beta_2}\cdot\binom{m_1+m_2-1}{m_2+\beta_2-\beta_1}\cdot\frac{m_2+\beta_2-\beta_1}{m_1+m_2-1}. \end{aligned} \tag{127}$$

The equation (127) implies (27) due to $\binom{n}{k}\cdot\frac{k}{n}=\binom{n-1}{k-1}$. □

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