HIGHER-ORDER DERIVATIVES OF FIRST-PASSAGE PERCOLATION WITH RESPECT TO THE ENVIRONMENT

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ABSTRACT. We introduce and study higher-order derivatives of first-passage percolation with respect to the environment. One of our main results is a generalization of the Benjamini–Kalai–Schramm–Talagrand variance bound, expressed in terms of the L^2-norms of these higher-order derivatives. We analyze the structure of these derivatives and compile a collection of related results. Several of these are sufficiently elementary to allow the use of algorithms and computer programs that automatically generate proofs of inequalities that would otherwise be intractable by manual methods.

1. INTRODUCTION

1.1. **Definition of the model.** The first-passage percolation model was introduced by Hammersley and Welsh in [23]. Fix two positive real numbers $a < b$ and a positive real number $p \in (0,1)$. Consider the graph whose vertices are elements of $\mathbb{Z}^d \cap [-2n,2n]^d$ (with $d \geq 2$), where two vertices $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)$ are connected by an edge if $|x_1-y_1|+\cdots+|x_d-y_d| = 1$.

Let W_n denote the set of edges of this graph. The sample space is defined as $\Omega_n = \{a,b\}^{W_n}$. Each edge e of the graph is independently assigned a passage time of either a or b, with probabilities $\mathbb{P}(a) = p$ and $\mathbb{P}(b) = 1-p$.

For a fixed environment $\omega \in \Omega_n$ and a path γ consisting of adjacent edges, the passage time $T(\gamma,\omega)$ is defined as the sum of the values assigned to the edges of γ. For two fixed vertices u and v, the passage time $f(u,v,\omega)$ is the random variable defined as the minimum of $T(\gamma,\omega)$ over all paths γ connecting u to v.

When v is fixed, we will also use the notation $f_n(\omega)$, f_n, or simply f, in place of $f(0,nv,\omega)$.

For some of our results, we will only work on a simplified model from [12] that has more symmetry and fewer technical challenges: the percolation is considered on the torus $\mathbb{Z}_n^d$. We will use superscript τ and write $f_n^\tau(\omega)$, f_n^τ, or f^τ to emphasize when we are working on this simplified torus model. Formally, a d-dimensional torus is a graph whose vertices are elements of $\mathbb{Z}_n^d$ and two vertices u and v are connected by an edge if and only if there is a coordinate $k \in \{1,2,\dots,d\}$ such that $u_k - v_k \equiv \pm 1 \pmod{n}$. The set of admissible paths Γ consists of all paths that wrap around the torus in the direction x_1. Hence, the path $(s,v_1,\dots,v_m,e)$ belongs to the set Γ if it is a path in the graph and if the starting and ending vertices s and e have all the coordinates the same except for the first coordinate. Their first coordinates are $s_1 = 0$ and $e_1 = n-1$. The random variable f_n^τ is defined as the minimum of the passage times among all paths γ that consist of adjacent edges and that wrap around the torus exactly once in the direction x_1. The function f_n^τ is defined as

$$f_n^\tau(\omega) \quad = \quad \min\{T(\gamma,\omega) : \gamma \in \Gamma\} \tag{1}$$

A path γ is called *geodesic* if the minimum $f_n(\omega)$ (or f_n^τ, depending on which problem we are studying) is attained at γ, i.e. if $f_n(\omega) = T(\gamma,\omega)$.

1.2. **Definition of environment derivatives.** If we denote by W_n the set of all edges, then the sample space is $\Omega_n = \{a,b\}^{W_n}$. We will often omit the subscript n when there is no danger of confusion. For each edge j and each $\omega \in \Omega$, we define $\sigma_j^a(\omega)$ as the outcome from Ω whose passage time over the edge j is changed from ω_j to a, regardless of what the original value ω_j was. The operation σ_j^b is defined in an analogous way. Formally, for $\delta \in \{a,b\}$, we define $\sigma_j^\delta : \Omega \to \Omega$ with

$$\left[\sigma_j^\delta(\omega)\right]_k = \begin{cases} \omega_k, & k \neq j, \\ \delta, & k = j. \end{cases} \tag{2}$$

If $\varphi : \Omega \to \mathbb{R}$ is any random variable, then the *first order environment derivative* $\partial_j \varphi$ is the random variable defined as

$$\partial_j \varphi = \varphi \circ \sigma_j^b - \varphi \circ \sigma_j^a. \tag{3}$$

For two distinct edges i and j, we will give the name *second order environment derivative* to the quantity $\partial_i \partial_j \varphi$. In general, if S is a non-empty subset of W, the operator $\partial_S \varphi$ is defined recursively as

$$\partial_S \varphi = \partial_{S\setminus\{j\}}\left(\partial_j \varphi\right), \tag{4}$$

where j is an arbitrary element of S. The definition (4) is independent on the choice of j, since a simple induction can be used to prove that for $S = \{s_1, \dots, s_m\}$, the following holds

$$\partial_S \varphi = \sum_{\theta_1 \in \{a,b\}} \cdots \sum_{\theta_m \in \{a,b\}} (-1)^{\mathbf{1}_a(\theta_1)+\cdots+\mathbf{1}_a(\theta_m)} \varphi \circ \sigma_{s_1}^{\theta_1} \circ \cdots \circ \sigma_{s_m}^{\theta_m}. \tag{5}$$

The function $\mathbf{1}_a : \{a,b\} \to \{0,1\}$ in (5) assigns the value 1 to a and 0 to b.

1.3. **Variance bounds.** The variances of f_n and f_n^τ can be bounded in terms of L^2 norms of environment derivatives. The results of Talagrand [34] and Benjamini, Kalai, and Schramm [12] show how inequalities with first order environment derivatives lead to bounds on the variance of the form $\mathrm{var}(f_n) \le C \cdot \frac{n}{\log n}$ and $\mathrm{var}(f_n^\tau) \le C \cdot \frac{n}{\log n}$, for some constant C. We will prove the following generalization of Talagrand's inequality.

Theorem 1. *Let f be a random variable on Ω. For every integer $k \ge 1$, there exists a real constant C and an integer n_0 such that for $n \ge n_0$, the following inequality holds*

$$\begin{aligned} var(f) \le & \sum_{M\subseteq W, 1\le |M| < k} (p(1-p))^{|M|} \left(\mathbb{E}\left[\partial_M f\right]\right)^2 \\ & + C \cdot \sum_{M \subseteq W, |M| = k, \|\partial_M f\|_1 \neq 0} \frac{\|\partial_M f\|_2^2}{1 + \left(\log \frac{\|\partial_M f\|_2}{\|\partial_M f\|_1}\right)^k}, \end{aligned} \tag{6}$$

where $\|g\|_p$ is the L^p-norm of the function g defined as

$$\|g\|_p = \left(\int_\Omega |g|^p \, d\mathbb{P}\right)^{1/p} = \left(\mathbb{E}[|g|^p]\right)^{1/p}.$$

For $k = 1$, the inequality (6) turns into Talagrand's Theorem 1.5 from [34].

The pair of edges (i,j) is called *convoluted* on the outcome $\omega \in \Omega$, if $\partial_i \partial_j f(\omega) \neq 0$. The following consequence provides an upper bound for the variance in terms of the expected number of convoluted pairs in the torus model.

Corollary 1. *If N_2 is the random variable that represents number of convoluted pairs of edges, then there exist constants C and $\hat{C}$ and an integer n_0 such that for $n \geq n_0$, the variance of f^τ on torus satisfies*

$$var(f^\tau) \quad \leq \quad C\frac{\sum_{|M|=2}\|\partial_M f^\tau\|_2^2}{(\log n)^2} \leq \hat{C}\frac{\mathbb{E}[N_2]}{(\log n)^2}. \tag{7}$$

The conjectured upper bound for the variance in first-passage percolation is $C \cdot n^{2\chi}$, where χ is an exponent that depends on the dimension. Current predictions suggest that χ is $\frac{1}{3}$ in the two-dimensional case [6]. A bound of $\chi \leq \frac{1}{2}$ was established by Kesten in 1993 [27], and to date, there is no formal proof that χ is strictly less than $\frac{1}{2}$. In dimensions higher than two, even conjectural values for χ remain unclear. According to [6], it is widely believed that χ remains strictly positive in all dimensions, though it tends to 0 as $d \to \infty$.

It is worth noting that the value $\chi = \frac{1}{3}$ was rigorously established by Johansson in a related model known as the totally asymmetric simple exclusion process (TASEP) [26]. The TASEP model belongs to a class of exactly solvable models that can be analyzed using techniques from random matrix theory. In this setting, a central limit theorem has been proven, with the limiting distribution given by the Tracy-Widom distribution for the largest eigenvalue [35].

The best current variance bound for first-passage percolation is $C \cdot \frac{n}{\log n}$. It is obtained by Benjamini, Kalai, and Schramm [12]. Their approach relies on Talagrand's inequality [34]. After applying the inequality, they use symmetries of the first-passage percolation models.

Our Theorem 1 generalizes Talagrand's inequality in the sense that the latter becomes a special case when $k = 1$. Moreover, in the torus model, we improve the denominator to $(\log n)^2$ in the Corollary 1, at the cost of introducing the term $\mathbb{E}[N_2]$ in the numerator. Currently, we are unable bound $\mathbb{E}[N_2]$ by n; hence, our result does not improve upon the best-known bound of $\frac{n}{\log n}$. We conjecture that $\mathbb{E}[N_2]$ and the L^2 norms $\|\partial_M f\|_2$ are small–especially in dimensions $d \geq 3$, where the decay could potentially be exponentially fast. However, these quantities remain difficult to analyze at present.

A complete understanding of the environment derivatives is equivalent to a complete understanding of the variance. We will see later in (53) that

$$\mathrm{var}(f) \quad = \quad \sum_{M\subseteq W, M\neq\emptyset} (p(1-p))^{|M|} \left(\mathbb{E}\left[\partial_M f\right]\right)^2.$$

The equation above is not particularly surprising–it is the Parseval's identity for the Fourier expansion of the variance in which the coefficients are expressed in terms of environment derivatives. Talagrand, as well as Benjamini, Kalai, and Schramm, have previously employed the Fourier expansion of variance, but skillfully avoided dealing with the coefficients directly by relying on clever bounding techniques.

Thus, the ultimate goal is to control the L^2-norms of the environment derivatives $\partial_M f$, as these norms are directly tied to the variance. At present, however, we are unable to effectively bound these L^2-norms.

This paper makes progress in analyzing the environment derivatives by establishing certain almost sure bounds. It also derives algebraic results concerning these derivatives, which allow us to bypass intuitive representations–representations that become increasingly difficult to construct when the sets M contain more than a few elements.

1.4. **Almost sure bounds on environment derivatives.** We start with definitions of two sequences. The first one will be the sequence $(\mathscr{U}_1, \mathscr{U}_2, \dots)$ of the most optimal upper bounds. The number $\mathscr{U}_k$ is defined as the most optimal upper bound on the k-th order environment derivative, i.e.

$$\mathscr{U}_k = \frac{1}{b-a} \max\left\{\partial_S f_n(\omega) : n \in \mathbb{N}, S \subseteq W_n, |S| = k, \omega \in \Omega_n\right\}. \tag{8}$$

The sequence $(\mathscr{L}_1, \mathscr{L}_2, \dots)$ of most optimal lower bounds is defined in an analogous way.

$$\mathscr{L}_k = \frac{1}{b-a} \min\left\{\partial_S f_n(\omega) : n \in \mathbb{N}, S \subseteq W_n, |S| = k, \omega \in \Omega_n\right\}. \tag{9}$$

The re-scaling factor $b-a$ is included to make the numbers $\mathscr{U}_k$ and $\mathscr{L}_k$ constant and independent on a and b.

Theorem 2. *The first four values of $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ are given in the table below.*

k	1	2	3	4
$\mathscr{U}_k$	1	1	2	3
$\mathscr{L}_k$	0	-1	-1	-2

(10)

Theorem 3. *The sequences $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ satisfy*

$$\mathscr{U}_{k+1} \leq \mathscr{U}_k - \mathscr{L}_k \quad \text{and} \quad \mathscr{L}_{k+1} \geq \mathscr{L}_k - \mathscr{U}_k, \tag{11}$$

for all $k \geq 1$. Moreover, for all $k \geq 2$,

$$\mathscr{U}_k \leq 2^{k-2} \quad \text{and} \quad |\mathscr{L}_k| \leq 2^{k-2}. \tag{12}$$

Also, there exists $k_0 \in \mathbb{N}$ such that for all integers $k \geq k_0$ the following holds

$$\mathscr{U}_k \geq \sqrt[4]{3}^k \quad \text{and} \quad |\mathscr{L}_k| \geq \sqrt[4]{3}^k. \tag{13}$$

Theorem 3 implies that $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ grow exponentially in k. The bounds (12) and (13) are somewhat generous, and they can be improved, for example, by using (11). In this paper, we focused on obtaining the precise values for $\mathscr{U}_k$ and $\mathscr{L}_k$ for small values of k. The bounds (12) and (13) are simple enough, so we decided to leave them as they are and point out that substantial improvements are very likely to happen in the future.

At the moment, the precise values of the elements of the sequences $(\mathscr{U}_k)$ and $(\mathscr{L}_k)$ are difficult to determine for large k. The equalities $\mathscr{U}_1 = 1$ and $\mathscr{L}_1 = 0$ were trivial; $\mathscr{U}_2 = 1$, $\mathscr{U}_3 = 2$, and $\mathscr{L}_2 = -1$ follow from (11) and (12) without much effort.

The lower bound $\mathscr{L}_3 = -1$ was of moderate complexity and was derived in Theorem 13. The upper bound $\mathscr{U}_4 = 3$ then followed easily from (11) and $\mathscr{L}_3 = -1$.

The lower bound $\mathscr{L}_4 = -2$ was obtained via a computer-assisted proof, and this work is presented in [30]. We anticipate that future research will extend these techniques to produce computer-assisted proofs for higher-order bounds as well.

1.5. **Methods used in proofs and overview of literature.** The proof of Theorem 1 relies on the Beckner–Bonami inequality from [10] and [13], similar to Talagrand's original approach. In our proof, we clearly separate probabilistic components from algebraic manipulations and extend the variance decomposition to gain higher powers in the denominator. The logarithmic improvement in the denominator is more transparent in our presentation due to this clearer separation between probability and algebra.

We modified Talagrand's method by generalizing his operator Δ_i (denoted ρ_i in [12]). Talagrand's operator is defined as

$$\Delta_i f(\omega) = f(\sigma_i(\omega)) - f(\omega),$$

where $\sigma_i(\omega)$ denotes the environment in which the passage time over edge i is changed from its original value. Our first-order environment derivative ∂_i is defined as

$$\partial_i f(\omega) = f(\sigma_i^b(\omega)) - f(\sigma_i^a(\omega)).$$

This seemingly small change leads to significant improvements in clarity, particularly in identifying edges that belong to geodesics and those for which the environment derivatives are nonzero. In addition, the integration-by-parts formulas become much simpler with the operator ∂_i, as it is a more natural extension of the classical derivative than Δ_i. If the denominator $(b-a)$ were introduced to normalize the derivative, the ordering of terms would align with the numerator $f \circ \sigma_i^b - f \circ \sigma_i^a$ of $\partial_i f$.

We generalize this environment derivative to higher orders, which allows us to make a tradeoff after applying the Beckner–Bonami inequality. This tradeoff improves the denominator from $\log n$ to $(\log n)^k$, but at the cost of introducing the L^2-norms $\|\partial_M f\|_2^2$ of the higher-order environment derivatives into the numerator. As mentioned earlier, these L^2-norms are not easy to control. We hope that other researchers will explore the theory of environment derivatives further, as they show promise for deeper understanding and improved bounds.

When the edge passage times are supported on $\{a,b\}$, as in the model studied in this paper, geodesics may not be unique. It is expected that there will be numerous sufficiently disjoint geodesics, which would imply a small number of influential edges. This, in turn, could make it easier to obtain bounds on N_2. The study of geodesics has produced several important results and highly credible conjectures. Notably, as the size of the environment grows, at least two infinite geodesics are expected to emerge [24]. Infinite geodesics are also known to coalesce with high probability [3], [33], [28].

If the edge passage times are continuously distributed, geodesics are unique, and the event $A_i = \{\partial_i f \neq 0\}$ coincides with the event that edge i is essential–that is that is, every geodesic passes through i. Benjamini, Kalai, and Schramm studied the discrete case and encountered a major challenge: proving that the probability of A_i decays as $n^{-\xi}$. If the event A_i occurs, we say that the edge i is influential. The authors of [12] proposed a simpler problem: prove that $\mathbb{P}(A_i) \to 0$. This problem was resolved recently. In the continuous setting, we now have bounds of the form $\mathbb{P}(A_j) \leq Cn^{-\xi}$. The first such results appeared in [20], were strengthened in [1] (which removed differentiability assumptions), and culminated in polynomial bounds in [22].

Over the past 20 years, the Benjamini-Kalai-Schramm trick has been successfully used to bound variances in numerous problems, many now categorized as superconcentration problems [15] or part of the Kardar–Parisi–Zhang (KPZ) universality class [2], [17]. First-passage percolation models can also be viewed as extreme cases of random polymers in the zero-temperature limit [36].

In [11] and [19], the $\frac{n}{\log n}$ variance bound was extended to a large class of distributions. The exponent χ, discussed earlier, is called the fluctuation exponent. It is related to the transversal exponent ξ, defined as the number for which $C \cdot n^{\xi}$ is the maximal distance from the geodesic to the straight line between the starting and ending point. The exponents χ and ξ satisfy the KPZ scaling relation $\chi = 2\xi - 1$. The inequality $\chi \geq 2\xi - 1$ was proved in [31], while the reverse inequality $\chi \leq 2\xi - 1$ was first shown in [14], then generalized and simplified in [5]. These scaling exponents are closely tied to the asymptotic shape of the balls in the first-passage percolation metric; see [18] and [16].

The models we study in this paper are discrete. However, there have been successful generalizations to models where graphs consist of points scattered in Euclidean space [25].

Scaling relations and large deviation estimates have been established for both these spatial models and traditional lattice models in [8] and [9]. In a broad class of first-passage percolation models, the limit shape has been shown to be differentiable [7]. These problems become even more continuous when framed in terms of random Hamilton-Jacobi equations. For generalizations of the law of large numbers and central limit theorems in this context, see [32], [4], and [21]. Variance bounds of the order $\frac{n}{\log n}$ have also been obtained in this continuous PDE setting in [29].

2. Essential and influential edges

We will distinguish four categories to which an edge of the graph can belong. These categories may overlap but are conceptually distinct. Most edges will not belong to any of them.

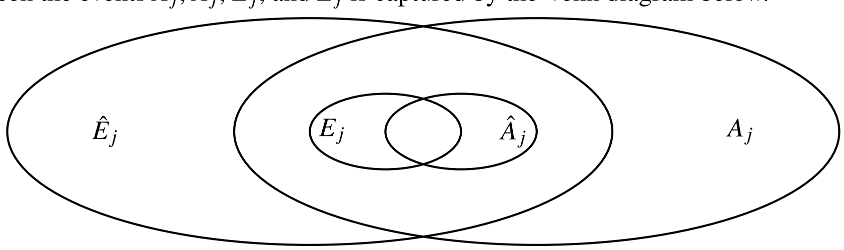
Definition 1. *An edge $j \in W_n$ is called* essential *on the environment ω if every geodesic passes through j. We will denote by E_j the event that the edge j is essential.*

Definition 2. *An edge $j \in W_n$ is called* semi-essential *on the environment ω if at least one geodesic passes through j. We will denote by $\hat{E}_j$ the event that the edge j is semi-essential.*

Definition 3. *An edge $j \in W_n$ is called* influential *if $\partial_j f(\omega) \neq 0$. We will denote by A_j the event that the edge j is influential.*

Definition 4. *An edge $j \in W_n$ is called* very influential *if $\partial_j f(\omega) = b - a$. We will denote by $\hat{A}_j$ the event that the edge j is very influential.*

Since the passage times across edges have a discrete distribution, there may be multiple geodesics between two fixed endpoints. We will show that, in general, the four categories defined above are distinct. Later in this section, we will prove that the general relationship between the events A_j, $\hat{A}_j$, E_j, and $\hat{E}_j$ is captured by the Venn diagram below.



The inclusion $E_j \subseteq A_j$ is the most important of all of the inclusions from the diagram. Although E_j is subset of A_j, we will prove later in Theorem 6 that the two events have comparable probabilities, i.e. $\mathbb{P}(A_j) \leq \mathbb{P}(E_j)/p$. It is natural to conjecture that all of the events E_j, A_j, $\hat{E}_j$ and $\hat{A}_j$ have comparable probabilities. We didn't need this full result in our paper, and the proof does not look obvious. Here is the formal conjecture.

Conjecture 1. *There exists a constant C independent on n such that*

$$\begin{aligned} \mathbb{P}(A_j) &\leq C \cdot \mathbb{P}(\hat{A}_j), \\ \mathbb{P}(\hat{E}_j) &\leq C \cdot \mathbb{P}(E_j), \quad \textit{and} \\ \mathbb{P}(\hat{E}_j) &\leq C \cdot \mathbb{P}(\hat{A}_j). \end{aligned}$$

Most research involving percolation models, where passage times have discrete distributions, has had to address these distinct categories of edges. Handling this distinction has often introduced technicalities that researchers needed to overcome. This section shows the similarities and differences among the various edge categories and summarizes their relationships. In our research, the clarifications provided by this section not only simplified the proofs but also served an additional purpose: they enabled us to identify fundamental relationships that were easiest to implement in computer-assisted proofs.

The results of this section apply to both f an f^τ; we will only present them for f. We will start with a proposition whose proof we will omit because it is straightforward.

Proposition 1. *For every $i \neq j$, every $\alpha, \beta \in \{a, b\}$, and every random variable φ,*

$$\begin{aligned} \sigma_i^\alpha \circ \sigma_i^\beta &= \sigma_i^\alpha; && (14)\\ \sigma_i^\alpha \circ \sigma_j^\beta &= \sigma_j^\beta \circ \sigma_i^\alpha; && (15)\\ (\partial_i \varphi) \circ \sigma_i^\alpha &= \partial_i \varphi; && (16)\\ \partial_i \partial_i \varphi &= 0; && (17)\\ \partial_i \partial_j \varphi &= \partial_j \partial_i \varphi; && (18)\\ \varphi \cdot 1_{\omega_i = \alpha} &= \varphi \circ \sigma_i^\alpha \cdot 1_{\omega_i = \alpha}. && (19) \end{aligned}$$

The next theorem is one of the few results that require a combinatorial analysis of geodesics. The obtained algebraic relationships among E_j, A_j, $\hat{E}_j$, and $\hat{A}_j$, together with some general results from set theory, imply all inclusions in the Venn diagram presented above.

Theorem 4. *The events E_j, A_j, $\hat{E}_j$, and $\hat{A}_j$ satisfy*

$$\begin{aligned} A_j &= (\sigma_j^a)^{-1}(E_j); && (20)\\ \hat{A}_j &= (\sigma_j^b)^{-1}(\hat{E}_j). && (21) \end{aligned}$$

Proof. We will first prove that $\{\partial_j f \neq 0\} \subseteq (\sigma_j^a)^{-1}(E_j)$. Assume the contrary, that there is ω that satisfies $\partial_j f(\omega) \neq 0$, but $\sigma_j^a(\omega) \notin E_j$. There is a geodesic γ on $\sigma_j^a(\omega)$ that does not pass through j. The value $f(\sigma_j^a(\omega))$ satisfies

$$f(\sigma_j^a(\omega)) = T(\gamma, \sigma_j^a(\omega)) = T(\gamma, \sigma_j^b(\omega)) \geq f(\sigma_j^b(\omega)).$$

The monotonicity of f in each coordinate implies $f(\sigma_j^a(\omega)) \leq f(\sigma_j^b(\omega))$. Hence, we obtained $f(\sigma_j^a(\omega)) = f(\sigma_j^b(\omega))$. This contradicts the assumption $\partial_j f(\omega) \neq 0$.

We will now prove that $(\sigma_j^a)^{-1}(E_j) \subseteq \{\partial_j f \neq 0\}$. Assume that $\omega \in (\sigma_j^a)^{-1}(E_j)$. We need to prove that $f(\sigma_j^b(\omega)) > f(\sigma_j^a(\omega))$. Let γ be a geodesic on $\sigma_j^b(\omega)$. There are two possibilities: $j \in \gamma$ and $j \notin \gamma$. In the case $j \in \gamma$, we have

$$\begin{aligned} f(\sigma_j^b(\omega)) &= T(\gamma, \sigma_j^b(\omega)) = (b-a) + T(\gamma, \sigma_j^a(\omega)) \geq (b-a) + f(\sigma_j^a(\omega))\\ &> f(\sigma_j^a(\omega)). \end{aligned}$$

In the case $j \notin \gamma$, the following holds

$$f(\sigma_j^b(\omega)) = T(\gamma, \sigma_j^b(\omega)) = T(\gamma, \sigma_j^a(\omega)).$$

However, since $\sigma_j^a(\omega) \in E_j$ and $j \notin \gamma$, the path γ cannot be a geodesic and $T(\gamma, \sigma_j^a(\omega)) > f(\sigma_j^a(\omega))$. We are allowed to conclude that $f(\sigma_j^b(\omega)) > f(\sigma_j^a(\omega))$. This completes the proof of (20).

We will now prove (21). Assume first that $\omega \in \left(\sigma_j^b\right)^{-1}(\hat{E}_j)$. Then, $\sigma_j^b(\omega) \in \hat{E}_j$, and there is a geodesic γ on $\sigma_j^b(\omega)$ that passes through j.

$$\begin{aligned} f(\sigma_j^b(\omega)) &= T(\gamma, \sigma_j^b(\omega)) = T(\gamma, \sigma_j^a(\omega)) + (b-a) \\ &\geq f(\sigma_j^a(\omega)) + b - a. \end{aligned}$$

It remains to observe that $f(\sigma_j^b(\omega)) \leq f(\sigma_j^a(\omega)) + b - a$. Therefore, $\omega \in \hat{A}_j$. We proved that $\left(\sigma_j^b\right)^{-1}(\hat{E}_j) \subseteq \hat{A}_j$.

Assume now that $\omega \in \hat{A}_j$. We need to prove that $\sigma_j^b(\omega) \in \hat{E}_j$. Let us consider $\sigma_j^a(\omega)$. If there is a geodesic δ on $\sigma_j^a(\omega)$ that does not pass through j, then

$$f(\sigma_j^a(\omega)) = T(\delta, \sigma_j^a(\omega)) = T(\delta, \sigma_j^b(\omega)) \geq f(\sigma_j^b(\omega)),$$

which would imply that $\partial_j f(\omega) = 0$ and contradict the assumption $\omega \in \hat{A}_j$. Hence, every geodesic on $\sigma_j^a(\omega)$ must pass through j and $\sigma_j^a(\omega) \in E_j$. Let γ be a geodesic on $\sigma_j^a(\omega)$. Since we assumed that $\omega \in \hat{A}_j$, we have

$$\begin{aligned} f(\sigma_j^b(\omega)) &= f(\sigma_j^a(\omega)) + (b-a) = T(\gamma, \sigma_j^a(\omega)) + (b-a) \\ &= T(\gamma, \sigma_j^b(\omega)). \end{aligned}$$

This means that γ is a geodesic on $\sigma_j^b(\omega)$. Since γ passes through j, we proved that $\sigma_j^b(\omega) \in \hat{E}_j$. □

The next proposition will be frequently used by computer algorithms that are proving inequalities involving environment derivatives. It is a simple consequence of equalities (20) and (21).

Proposition 2. *The following two implications hold for every $\omega \in \Omega$.*

(a) *If $\sigma_j^a(\omega) \in E_j^C$, then $f(\sigma_j^b(\omega)) = f(\sigma_j^a(\omega))$;*
(b) *If $\sigma_j^b(\omega) \in \hat{E}_j$, then $f(\sigma_j^b(\omega)) = f(\sigma_j^a(\omega)) + (b-a)$.*

Proof. Part (a) follows directly from (20). If we assume $\sigma_j^a(\omega) \in E_j^C$, then

$$\omega \in (\sigma_j^a)^{-1}(E_j^C) = \left((\sigma_j^a)^{-1}(E_j)\right)^C = A_j^C.$$

Similarly, part (b) is a direct consequence of (21). □

The functions σ_i^a and σ_i^b are idempotent (a function $\psi : R \to R$ is idempotent if $\psi \circ \psi = \psi$) and they satisfy $\sigma_i^a \circ \sigma_i^b = \sigma_i^a$ and $\sigma_i^b \circ \sigma_i^a = \sigma_i^b$. These algebraic properties, together with $E_j \subseteq \hat{E}_j$, $\hat{A}_j \subseteq A_j$, (20), and (21) will have the following algebraic consequence: $E_j \subseteq A_j$ and $\hat{A}_j \subseteq \hat{E}_j$. These inclusions (and several more results) will follow from the following general properties of images and pre-images of idempotent functions, whose proofs are left for the Appendix A.

Proposition 3. *If $\psi : R \to R$ is an idempotent function, then the set of its fixed points is equal to its range, i.e.*

$$\psi(R) = \{x \in R : \psi(x) = x\}. \tag{22}$$

Proposition 4. *Assume that $\psi : R \to R$ and $\xi : R \to R$ are two idempotent functions that satisfy $\psi \circ \xi = \psi$. If Q is any subset of R, then*

$$\psi\left(\psi^{-1}(Q) \cap \xi(R)\right) = \psi\left(\psi^{-1}(Q)\right) = Q \cap \psi(R) = \psi^{-1}(Q) \cap \psi(R). \tag{23}$$

Proposition 5. *Assume that $\psi^a : R \to R$ and $\psi^b : R \to R$ are two idempotent functions that satisfy $\psi^a(R) \cup \psi^b(R) = R$. If E and $\hat{E}$ are two subsets of R that satisfy $E \subseteq \hat{E}$ and $(\psi^b)^{-1}(\hat{E}) \subseteq (\psi^a)^{-1}(E)$, then*

$$E \subseteq (\psi^a)^{-1}(E), \tag{24}$$

$$(\psi^b)^{-1}(\hat{E}) \subseteq \hat{E}. \tag{25}$$

The equation (22) applied to σ^a_j and σ^b_j transforms into

$$\sigma^a_j(\Omega) = \{\omega_j = a\} \quad \text{and} \quad \sigma^b_j(\Omega) = \{\omega_j = b\}. \tag{26}$$

We will apply the equation (23) to idempotent functions σ^a_j and σ^b_j that satisfy $\sigma^a_j \circ \sigma^b_j = \sigma^a_j$ and $\sigma^b_j \circ \sigma^a_j = \sigma^b_j$.

Proposition 6. *For every $j \in W$, the events E_j, $\hat{E}_j$, A_j, and $\hat{A}_j$ satisfy*

$$\sigma^a_j(A_j) = \sigma^a_j(A_j \cap \{\omega_j = b\}) = A_j \cap \{\omega_j = a\} = E_j \cap \{\omega_j = a\}; \tag{27}$$

$$\sigma^b_j(\hat{A}_j) = \sigma^b_j(\hat{A}_j \cap \{\omega_j = a\}) = \hat{A}_j \cap \{\omega_j = b\} = \hat{E}_j \cap \{\omega_j = b\}. \tag{28}$$

Proof. The given equalities are direct consequences of (14), (20), (21), (23), and (26). $\square$

Theorem 5. *For every $j \in W$, the events E_j, $\hat{E}_j$, A_j, and $\hat{A}_j$ satisfy*

$$E_j \subseteq \hat{E}_j; \quad \hat{A}_j \subseteq A_j; \tag{29}$$

$$E_j \subseteq A_j; \quad \text{and} \tag{30}$$

$$\hat{A}_j \subseteq \hat{E}_j. \tag{31}$$

Proof. The inclusions (29) are obvious, while the inclusions (30) and (31) follow from (24) and (25). $\square$

In some special cases (one of which is $a, b \in \mathbb{N}$, $b = a+1$), the events $\hat{A}_j$ and A_j are the same, and the relationships between E_j, A_j, and $\hat{E}_j$ are simpler.

Proposition 7. *Assume that the real numbers a and b are such that there are no integers k_a and k_b for which $ak_a + bk_b$ belongs to the open interval $(0, b-a)$. Then,*

$$E_j \subseteq A_j = \hat{A}_j \subseteq \hat{E}_j.$$

Proof. This is a trivial consequence of $E_j \subseteq A_j$, $A_j = \hat{A}_j$, and $\hat{A}_j \subseteq E_j$. $\square$

Proposition 8. *For sufficiently large n, the following sets are non-empty: $\hat{E}_j \setminus A_j$, $\hat{A}_j \setminus E_j$, and $A_j \setminus E_j$. If there exist integers k_a and k_b such that $ak_a + bk_b \in (0, b-a)$, then for sufficiently large n, the sets $A_j \setminus \hat{E}_j$ and $E_j \setminus \hat{A}_j$ are non-empty.*

Proof. There are trivial examples that establish $\hat{E}_j \setminus A_j \neq \emptyset$ and $\hat{A}_j \setminus E_j \neq \emptyset$. We will construct them on the torus model. The examples can be easily extended to the general first-passage percolation. Let ω_a be the environment that assigns the value a to every edge. It is easy to prove that $\omega_a \in \hat{E}_j \setminus A_j$, for every edge j that connects the vertex $\overrightarrow{x}$ with the vertex $\overrightarrow{y}$ and satisfies $x_1 - y_1 = \pm 1$. Take now the environment ω_b whose all edges are assigned the value b. Then, every edge between $\overrightarrow{x}$ and $\overrightarrow{y}$ that satisfies $x_1 - y_1 = \pm 1$ is very influential. None of the edges is essential, hence $\omega_b \in \hat{A}_j \setminus E_j$. These two trivial examples ω_a and ω_b showed that $\hat{E}_j \setminus A_j$ and $\hat{A}_j \setminus E_j$ are non-empty.

In Section 4, we will prove that the remaining sets are non-empty. The relation $A_j \setminus E_j \neq \emptyset$ is proved in Proposition 20. The Proposition 21 proves that $A_j \setminus \hat{E}_j$ and $E_j \setminus \hat{A}_j$ are non-empty if there exist integers k_a and k_b such that $ak_a + bk_b \in (0, b-a)$. $\square$

Proposition 9. *Assume that $\omega \in E_j$. A path γ is a geodesic on ω if and only if it is a geodesic on $\sigma_j^a(\omega)$.*

Proof. First we will prove that every geodesic on ω is also a geodesic on $\sigma_j^a(\omega)$. The case $\omega_j = a$ is trivial. Assume that $\omega_j = b$. Let μ be a geodesic on ω. Since $\omega \in E_j$, we must have

$$f(\omega) \quad = \quad T(\mu,\omega) = T(\mu,\sigma_j^a(\omega)) + (b-a).$$

We obtained that the equation $T(\mu,\sigma_j^a(\omega)) = f(\omega) - (b-a)$ holds for every geodesic μ on ω. If ν is any other path that is not a geodesic on ω, we must have $T(\nu,\omega) > f(\omega)$. We would also have

$$T(\nu,\sigma_j^a(\omega)) \geq T(\nu,\omega) - (b-a) > f(\omega) - (b-a) = T(\mu,\sigma_j^a(\omega)).$$

We have made the following conclusion: If ν is not a geodesic on ω, then ν is also not a geodesic on $\sigma_j^a(\omega)$. In addition, all geodesics on ω have the same passage time on $\sigma_j^a(\omega)$. This implies that all geodesics on ω are geodesics on $\sigma_j^a(\omega)$.

Let us now prove that every geodesic on $\sigma_j^a(\omega)$ is also a geodesic on ω. It suffices to prove this for $\omega \in E_j \cap \{\omega_j = b\}$. Assume the contrary, that there is a geodesic γ on $\sigma_j^a(\omega)$ that is not a geodesic on ω. Since

$$\sigma_j^a(E_j) \subseteq \sigma_j^a(A_j) = E_j \cap \{\omega_j = a\} \subseteq E_j,$$

we must have $\sigma_j^a(\omega) \in E_j$. Therefore, the path γ must pass through j on $\sigma_j^a(\omega)$. Therefore,

$$f(\sigma_j^a(\omega)) \quad = \quad T(\gamma,\sigma_j^a(\omega)) = T(\gamma,\omega) - (b-a). \tag{32}$$

Since γ is not a geodesic on ω, there must be a path δ which is a geodesic and for which $T(\gamma,\omega)$ is strictly larger than $T(\delta,\omega)$. However, $\omega \in E_j$ by the assumption. Therefore, δ passes through j and $T(\delta,\omega) = T(\delta,\sigma_j^a(\omega)) + (b-a)$. From (32) we obtain

$$\begin{aligned} f(\sigma_j^a(\omega)) &= T(\gamma,\omega) - (b-a) \\ &> T(\delta,\omega) - (b-a) \\ &= T(\delta,\sigma_j^a(\omega)). \end{aligned}$$

This is a contradiction, because the value f must be smaller than or equal to the cost over the path δ on the envifornment $\sigma_j^a(\omega)$. □

For $V \subseteq W$, define $E_V = \bigcap_{j\in V} E_j$.

Proposition 10. *For every $j \in V$, the following holds*

$$\sigma_j^a(E_V) \quad = \quad E_V \cap \{\omega_j = a\}. \tag{33}$$

Proof. The inclusion $\supseteq$ is obvious: If $\omega \in E_V$ and $\omega_j = a$, then $\sigma_j^a(\omega) = \omega$. Therefore, the environment ω is the image of ω under σ_j^a. That makes ω an element of $\sigma_j^a(E_V)$.

We need to prove that $\sigma_j^a(E_V) \subseteq E_V$. Take $\zeta \in \sigma_j^a(E_V)$. There exists $\omega \in E_V$ such that $\zeta = \sigma_j^a(\omega)$. Since $\omega \in E_V \subseteq E_j$, we can apply Proposition 9. Every geodesic on ζ must be a geodesic on ω. However, $\omega \in E_V$. Every geodesic on ω must pass through all of the vertices of V. All of the geodesics on ζ must satisfy the same condition. Thus, $\zeta \in E_V$. □

If $\overrightarrow{\alpha} \in \{a,b\}^m$ and $\overrightarrow{\nu} \in W^m$, define $\sigma_{\overrightarrow{\nu}}^{\overrightarrow{\alpha}} : \Omega \to \Omega$ as

$$\sigma_{\overrightarrow{\nu}}^{\overrightarrow{\alpha}} \quad = \quad \sigma_{\nu_1}^{\alpha_1} \circ \cdots \circ \sigma_{\nu_m}^{\alpha_m}, \tag{34}$$

where $\alpha_1, \ldots, \alpha_m$ are the components of $\overrightarrow{\alpha}$ and $\nu_1, \ldots, \nu_m$ are the components of $\overrightarrow{\nu}$.

The event $I_{\overrightarrow{v},\overrightarrow{\alpha}}$ is defined as

$$I_{\overrightarrow{v},\overrightarrow{\alpha}} = \sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}(\Omega) = \left\{\omega\in\Omega : \omega_{v_1}=\alpha_1,\dots,\omega_{v_k}=\alpha_k\right\}. \tag{35}$$

Proposition 11. *For every event A, every vector $\overrightarrow{v}\in W^k$ that has all distinct components, and every two vectors $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ from $\{a,b\}^m$, the following holds*

$$\mathbb{P}\left(A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}\right) = \frac{\mathbb{P}(\overrightarrow{\beta})}{\mathbb{P}(\overrightarrow{\alpha})}\mathbb{P}\left(\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}\left(A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}\right)\right). \tag{36}$$

Proof. Take $\omega\in\Omega$ and fix $\overrightarrow{v}$. When we remove the components of ω whose indices appear in the vector $\overrightarrow{v}$, we obtain a shorter sequence $R_{\overrightarrow{v}}(\omega)$ that is an element of $\Omega_{\overrightarrow{v}} = \{a,b\}^{W\setminus\{\overrightarrow{v}\}}$. Let us denote by $\mathbb{P}_{\overrightarrow{v}}$ the induced probability measure on $\Omega_{\overrightarrow{v}}$.

$$\begin{aligned}\mathbb{P}\left(A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}\right) &= \sum_{\omega\in A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}}\mathbb{P}(\omega) = \sum_{\omega\in A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}}\mathbb{P}(\overrightarrow{\beta})\mathbb{P}_{\overrightarrow{v}}(R_{\overrightarrow{v}}(\omega))\\ &= \frac{\mathbb{P}(\overrightarrow{\beta})}{\mathbb{P}(\overrightarrow{\alpha})}\sum_{\omega\in A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}}\mathbb{P}(\overrightarrow{\alpha})\mathbb{P}_{\overrightarrow{v}}(R_{\overrightarrow{v}}\omega).\end{aligned}$$

Observe that $\mathbb{P}(\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}(\omega)) = \mathbb{P}(\overrightarrow{\alpha})\mathbb{P}_{\overrightarrow{v}}(R_{\overrightarrow{v}}(\omega))$. Therefore,

$$\mathbb{P}\left(A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}\right) = \frac{\mathbb{P}(\overrightarrow{\beta})}{\mathbb{P}(\overrightarrow{\alpha})}\sum_{\omega\in A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}}\mathbb{P}(\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}\omega).$$

Since $\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}$ is a bijection from $A\cap I_{\overrightarrow{v},\overrightarrow{\beta}}$ to $\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}(A\cap I_{\overrightarrow{v},\overrightarrow{\beta}})$, we can use the substitution $\zeta=\sigma_{\overrightarrow{v}}^{\overrightarrow{\alpha}}(\omega)$ in the last summation and obtain the equation (36) . □

Let us state a special case of the previous proposition in which the dimensions of vectors are all 1.

Proposition 12. *For every event A and every pair $(\alpha,\beta)\in\{a,b\}^2$ and every $j\in W$, the following holds*

$$\mathbb{P}\left(A\cap\left\{\omega_j=\beta\right\}\right) = \frac{\mathbb{P}(\beta)}{\mathbb{P}(\alpha)}\mathbb{P}\left(\sigma_j^{\alpha}\left(A\cap\left\{\omega_j=\beta\right\}\right)\right). \tag{37}$$

We now use (37) and (36) to prove the following two theorems.

Theorem 6. *For every edge $i\in W_n$, the probabilities of the events A_i and E_i satisfy or every $i\in W$ we have*

$$\mathbb{P}(A_i) \leq \frac{1}{p}\mathbb{P}(E_i). \tag{38}$$

Proof. Using (27) we obtain

$$\begin{aligned}\mathbb{P}(A_i) &= \mathbb{P}(A_i\cap\{\omega_i=a\})+\mathbb{P}(A_i\cap\{\omega_i=b\})\\ &= \mathbb{P}(E_i\cap\{\omega_i=a\})+\mathbb{P}(A_i\cap\{\omega_i=b\})\\ &\leq \mathbb{P}(E_i)+\mathbb{P}(A_i\cap\{\omega_i=b\}).\end{aligned} \tag{39}$$

We now use (37) and (27) to bound the second term on the right-hand side of (39).

$$\begin{aligned} \mathbb{P}\left(A_i \cap \{\omega_i = b\}\right) &= \frac{\mathbb{P}(b)}{\mathbb{P}(a)} \mathbb{P}\left(\sigma_i^a \left(A_i \cap \{\omega_i = b\}\right)\right) = \frac{\mathbb{P}(b)}{\mathbb{P}(a)} \mathbb{P}\left(E_i \cap \{\omega_i = a\}\right) \\ &\leq \frac{\mathbb{P}(b)}{\mathbb{P}(a)} \mathbb{P}\left(E_i\right). \end{aligned} \tag{40}$$

The inequality in (38) is now a direct consequence of (39) and (40). □

Theorem 7. *Assume that $\overrightarrow{v} \in W^m$ has all distinct components. Assume that j is not a component of $\overrightarrow{v}$. Assume that $\overrightarrow{\gamma}$ and $\overrightarrow{\delta}$ are elements of $\{a,b\}^m$. Then,*

$$\mathbb{P}\left(\left\{\partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}} \neq 0\right\} \cap I_{\overrightarrow{v},\overrightarrow{\delta}}\right) \leq \frac{\mathbb{P}(\overrightarrow{\delta})}{\mathbb{P}(\overrightarrow{\gamma})} \mathbb{P}(A_j \cap I_{\overrightarrow{v},\overrightarrow{\gamma}}). \tag{41}$$

Proof. We first use (36) to derive

$$\mathbb{P}\left(\left\{\partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}} \neq 0\right\} \cap I_{\overrightarrow{v},\overrightarrow{\delta}}\right) = \frac{\mathbb{P}(\overrightarrow{\delta})}{\mathbb{P}(\overrightarrow{\gamma})} \mathbb{P}\left(\sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}}\left(\left\{\partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}} \neq 0\right\} \cap I_{\overrightarrow{v},\overrightarrow{\delta}}\right)\right).$$

It suffices to prove that

$$\sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}}\left(\left\{\partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}} \neq 0\right\} \cap I_{\overrightarrow{v},\overrightarrow{\delta}}\right) \subseteq A_j \cap I_{\overrightarrow{v},\overrightarrow{\gamma}}. \tag{42}$$

Assume that $\zeta \in \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}}\left(\left\{\partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}} \neq 0\right\} \cap I_{\overrightarrow{v},\overrightarrow{\delta}}\right)$. There exists $\omega \in \left\{\partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}} \neq 0\right\} \cap I_{\overrightarrow{v},\overrightarrow{\delta}}$ such that $\zeta = \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}}(\omega)$. Clearly, $\zeta \in I_{\overrightarrow{v},\overrightarrow{\gamma}}$. In order to prove that $\zeta \in A_j$, we need to prove that $\partial_j f(\zeta) \neq 0$. However, $\partial_j f(\zeta) = \partial_j f(\sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}}(\omega)) = \partial_j f \circ \sigma_{\overrightarrow{v}}^{\overrightarrow{\gamma}}(\omega) \neq 0$ by our choice of ω. □

3. VARIANCE DECOMPOSITION

3.1. **Integration by parts.** We will first establish theorems that hold for general random variables, not only first-passage percolation times. Theorems 8 and 9 are analogous to integration by parts formulas from calculus.

Proposition 13. *For every random variable φ on Ω, its expected value $\mathbb{E}[\varphi]$ can be evaluated using the equation*

$$\mathbb{E}[\varphi] = p\mathbb{E}[\varphi \circ \sigma_i^a] + (1-p)\mathbb{E}[\varphi \circ \sigma_i^b]. \tag{43}$$

Proof. Let us denote by $\mathbb{P}_i$ the product measure on $\{a,b\}^{W\setminus\{i\}}$ defined as

$$\mathbb{P}_i(\omega) = \prod_{k\in W\setminus\{i\}} p^{\mathbf{1}_a(\omega_k)}(1-p)^{1-\mathbf{1}_a(\omega_k)}. \tag{44}$$

Let $\mathbb{E}_i$ be the corresponding expected value. We use (19) to obtain

$$\begin{aligned} \mathbb{E}[\varphi] &= \mathbb{E}\left[\varphi \cdot 1_{\omega_i=a}\right] + \mathbb{E}\left[\varphi \cdot 1_{\omega_i=b}\right] \\ &= \mathbb{E}\left[\varphi \circ \sigma_i^a \cdot 1_{\omega_i=a}\right] + \mathbb{E}\left[\varphi \circ \sigma_i^b \cdot 1_{\omega_i=b}\right] \\ &= p\mathbb{E}_i\left[\varphi \circ \sigma_i^a\right] + (1-p)\mathbb{E}_i\left[\varphi \circ \sigma_i^b\right]. \end{aligned} \tag{45}$$

We can apply (45) to $\varphi\circ\sigma_i^a$ and use (14).

$$\begin{aligned}\mathbb{E}[\varphi\circ\sigma_i^a] &= p\mathbb{E}_i\left[\varphi\circ\sigma_i^a\circ\sigma_i^a\right]+(1-p)\mathbb{E}_i\left[\varphi\circ\sigma_i^a\circ\sigma_i^b\right]\\ &= p\mathbb{E}_i\left[\varphi\circ\sigma_i^a\right]+(1-p)\mathbb{E}_i\left[\varphi\circ\sigma_i^a\right]\\ &= \mathbb{E}_i\left[\varphi\circ\sigma_i^a\right].\end{aligned}$$

In an analogous way we derive the equality $\mathbb{E}\left[\varphi\circ\sigma_i^b\right]=\mathbb{E}_i\left[\varphi\circ\sigma_i^b\right]$. The equation (45) becomes (43). □

For $\omega\in\Omega$ and $i\in W$, let us define

$$r_i(\omega) = \begin{cases}-\sqrt{\frac{1-p}{p}}, & \text{if } \omega_i=a,\\ \sqrt{\frac{p}{1-p}}, & \text{if } \omega_i=b.\end{cases} \tag{46}$$

For each i, we have $\mathbb{E}[r_i]=0$ and $\mathrm{var}(r_i)=1$. For $S\subseteq W$, we define

$$r_S(\omega) = \prod_{i\in S} r_i(\omega). \tag{47}$$

Due to independence, we have $\mathbb{E}[r_S]=0$ and $\mathrm{var}(S)=1$ whenever $S\neq\emptyset$. Also, if S and T are different sets, then their dot product $\mathbb{E}[r_Sr_T]$ must be equal to 0. Therefore, the functions $(r_S)_{S\subseteq W}$ form an orthonormal basis of $L^2(\Omega)$.

Theorem 8. *For every nonempty $S\subseteq W$ and every random variable φ, the expected value of φr_S satisfies*

$$\mathbb{E}[\varphi r_S] = \sqrt{p(1-p)}^{|S|}\,\mathbb{E}[\partial_S\varphi]. \tag{48}$$

Theorem 8 is a special case obtained by placing $S=T$ in the following more general result.

Theorem 9. *For every two sets T and S with $T\subseteq S\subseteq W$, and every random variable φ, the expected value of φr_S satisfies*

$$\mathbb{E}[\varphi r_S] = \sqrt{p(1-p)}^{|T|}\,\mathbb{E}\left[(\partial_T\varphi)\cdot r_{S\setminus T}\right]. \tag{49}$$

Proof. Let us first consider the case $T=\{i\}$. An application of (43) results in

$$\begin{aligned}\mathbb{E}[\varphi r_S] &= p\mathbb{E}\left[\varphi r_S\circ\sigma_i^a\right]+(1-p)\mathbb{E}\left[\varphi r_S\circ\sigma_i^b\right]\\ &= p\mathbb{E}\left[(\varphi\circ\sigma_i^a)(r_S\circ\sigma_i^a)\right]+(1-p)\mathbb{E}\left[\left(\varphi\circ\sigma_i^b\right)\left(r_S\circ\sigma_i^b\right)\right]\\ &= -p\mathbb{E}\left[(\varphi\circ\sigma_i^a)\cdot\sqrt{\frac{1-p}{p}}r_{S\setminus\{i\}}\right]+(1-p)\mathbb{E}\left[\left(\varphi\circ\sigma_i^b\right)\cdot\sqrt{\frac{p}{1-p}}r_{S\setminus\{i\}}\right]\\ &= \sqrt{p(1-p)}\mathbb{E}\left[\left(\varphi\circ\sigma_i^b-\varphi\circ\sigma_i^a\right)r_{S\setminus\{i\}}\right]\\ &= \sqrt{p(1-p)}\mathbb{E}\left[\partial_i\varphi r_{S\setminus\{i\}}\right].\end{aligned} \tag{50}$$

The general result (49) follows by a straightforward induction. □

Proposition 14. *Assume that $U\subseteq V\subseteq W$. If $\partial_U g(\omega)=0$ for all ω, then $\partial_V g(\omega)=0$ for all ω.*

Proof. Let $Q \subseteq W \setminus V$. Let us apply the formula (49) to $\varphi = \partial_U g$, $T = V \setminus U$, and $S = \{V \setminus U\} \cup Q$.

$$\begin{aligned} 0 &= \mathbb{E}\left[(\partial_U g) \cdot r_{\{V \setminus U\} \cup Q}\right] = \sqrt{p(1-p)}^{|V \setminus U|} \mathbb{E}\left[\left(\partial_{V \setminus U} \partial_U g\right) \cdot r_Q\right] \\ &= \sqrt{p(1-p)}^{|V \setminus U|} \mathbb{E}\left[(\partial_V g) \cdot r_Q\right]. \end{aligned} \tag{51}$$

The equation (51) holds for every $Q \subseteq W \setminus V$. If Q is a set that has non-trivial intersection with V, then $\mathbb{E}\left[(\partial_V g) \cdot r_Q\right]$ is equal to 0 because of (17). Hence, $\partial_V g$ is orthogonal to every element of the basis. Thus, $\partial_V g = 0$. □

3.2. **Generalization of Talagrand's inequality.** We will use the following variant of Beckner-Bonami inequality.

Theorem 10. *If g is an element of span$\{r_Q : |Q| \leq L\}$, then for each $q \geq 2$, there exists a constant $\alpha = \alpha(p,q) > 0$ such that*

$$\|g\|_2^2 \leq e^{\alpha L} \|g\|_{q'}^2,$$

where q' is the conjugate of q.

Proof. This result is proved in [34]. It is listed as the Proposition 2.2 on page 1580. □

We use Beckner-Bonami inequality to generalize Talagrand's theorem 1.5 from [34].

Proof of Theorem 1. The coefficients of f in base $(r_S)_{S \subseteq W}$ will be denoted by a_S. The coefficients satisfy

$$a_S = \mathbb{E}\left[f r_S\right] = \sqrt{p(1-p)}^{|S|} \mathbb{E}\left[\partial_S f\right]. \tag{52}$$

Except for the coefficient $a_\emptyset$, the random variables $f - \mathbb{E}[f]$ and f have the same coefficients. Therefore,

$$\mathrm{var}(f) = \sum_{S \subseteq W, S \neq \emptyset} (p(1-p))^{|S|} \left(\mathbb{E}\left[\partial_S f\right]\right)^2. \tag{53}$$

The variance is the sum of squares of all Fourier coefficients a_S^2. The sum will be decomposed into two sums: the lower sum and the higher sum.

$$\begin{aligned} \mathrm{var}(f) &= L_k(f) + H_k(f), \quad \text{where} \\ L_k(f) &= \sum_{1 \leq |S| < k} a_S^2, \qquad (54) \\ H_k(f) &= \sum_{|S| \geq k} a_S^2. \qquad (55) \end{aligned}$$

The lower sum will only undergo algebraic transformations. Equation (48) implies

$$a_S = \mathbb{E}\left[f r_S\right] = \sqrt{p(1-p)}^{|S|} \cdot \mathbb{E}\left[\partial_S f\right].$$

The lower sum $L_k(f)$ can now be written as

$$L_k(f) = \sum_{1 \leq |S| < k} (p(1-p))^{|S|} \left(\mathbb{E}\left[\partial_S f\right]\right)^2. \tag{56}$$

The higher sum will be first re-organized. Let $\mathscr{F}$ be the family of those subsets of W that have cardinality at least k. Let us introduce some notation that will simplify the writing. We will denote by $\mathscr{I}_T$ the set of all subsets of W that contain T, i.e.

$$\mathscr{I}_T = \{S \subseteq W : T \subseteq S\}.$$

In the case when T consists of a single element t, we may write $\mathscr{I}_t$ instead of $\mathscr{I}_{\{t\}}$.

$$\begin{aligned}
H_k(f) &= \sum_{S\in\mathscr{F}} a_S^2 = \sum_{S\in\mathscr{F}} a_S^2\cdot\frac{1}{|S|}\cdot\sum_{i_1\in W} 1_{i_1\in S} = \sum_{i_1\in W}\left(\sum_{S\in\mathscr{F}\cap\mathscr{I}_{i_1}}\frac{a_S^2}{|S|}\right)\\
&= \sum_{i_1\in W}\left(\sum_{S\in\mathscr{F}\cap\mathscr{I}_{i_1}}\frac{a_S^2}{|S|(|S|-1)}\sum_{i_2\in W\setminus\{i_1\}} 1_{i_2\in S}\right)\\
&= \sum_{i_1\in W}\left(\sum_{i_2\in W\setminus\{i_1\}}\left(\sum_{S\in\mathscr{F}\cap\mathscr{I}_{\{i_1,i_2\}}}\frac{a_S^2}{|S|(|S|-1)}\right)\right). \qquad (57)
\end{aligned}$$

We can now continue in the same way as in (57) until the number of indices becomes k. The summation (57) becomes

$$H_k(f) = \sum_{M\subseteq W,|M|=k} k!\cdot\left(\sum_{S\in\mathscr{F}\cap\mathscr{I}_M}\frac{a_S^2}{|S|(|S|-1)(|S|-2)\cdots(|S|-k+1)}\right). \qquad (58)$$

We now apply (49) with $\varphi = f$ and $T = M$ to obtain

$$a_S = \mathbb{E}\left[f\cdot r_S\right] = \sqrt{p(1-p)}^{|M|}\cdot\mathbb{E}\left[\partial_M f\cdot r_{S\setminus M}\right]. \qquad (59)$$

Equations (58) and (59) imply

$$H_k(f) = \sum_{M\subseteq W,|M|=k} k!\cdot\left(\sum_{S\in\mathscr{F}\cap\mathscr{I}_M}\frac{(p(1-p))^k\mathbb{E}\left[(\partial_M f)\,r_{S\setminus M}\right]^2}{|S|(|S|-1)(|S|-2)\cdots(|S|-k+1)}\right). \qquad (60)$$

For fixed M, let us denote by $\Sigma(M)$ the inner summation in (60). Formally,

$$\Sigma(M) = k!\cdot\left(\sum_{S\in\mathscr{F}\cap\mathscr{I}_M}\frac{(p(1-p))^k\mathbb{E}\left[(\partial_M f)\,r_{S\setminus M}\right]^2}{|S|(|S|-1)(|S|-2)\cdots(|S|-k+1)}\right). \qquad (61)$$

We will assume that the summation is restricted to the sets M for which $\|\partial_M f\|_1$ is non-zero. Since we are working with finite sample space, the L^1-norm is zero only when the function $\partial_M f$ is identically equal to 0. We split $\Sigma(M)$ into two groups: the summation H_M^- corresponding to sets of sizes smaller than L_M; and the summation H_M^+ corresponding to sets of sizes larger than or equal to L_M. The integer L_M will be determined later. If the number of elements of S is at least k, then the product of the numbers $|S|$, $|S|-1$, $\dots$, $|S|-k+1$ in the denominator is greater than or equal to $k!$. Hence,

$$\begin{aligned}
H_M^- &= k!\cdot\sum_{S\in\mathscr{I}_M,k\leq|S|<L_M}\frac{(p(1-p))^k\mathbb{E}\left[(\partial_M f)\,r_{S\setminus M}\right]^2}{|S|(|S|-1)(|S|-2)\cdots(|S|-k+1)}\\
&\leq (p(1-p))^k\left(\sum_{S\in\mathscr{I}_M,k\leq|S|<L_M}\mathbb{E}\left[(\partial_M f)\,r_{S\setminus M}\right]^2\right).
\end{aligned}$$

There is an obvious bijection between $\mathscr{I}_M$ and the subsets of $W\setminus M$. Hence, we can do the substitution $Q = S\setminus M$ and obtain the following bound for H_M^-.

$$H_M^- \leq (p(1-p))^k\sum_{Q\subseteq W\setminus M,|Q|<L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]^2.$$

We will now use (60) to find an upper bound for H_M^+. The cardinalities of sets S are now bigger than or equal to L_M. Therefore, the product of numbers $|S|$, $|S|-1$, $\dots$, $|S|-k+1$

in the denominator is greater than or equal to $L_M\cdot(L_M-1)\cdots(L_M-k+1)$ which is equal to $k!\cdot\binom{L_M}{k}$. Therefore, the sum H_M^+ satisfies

$$\begin{aligned} H_M^+ &= k!\sum_{S\in\mathscr{I}_M,|S|\geq L_M}\frac{(p(1-p))^k\mathbb{E}\left[(\partial_M f)\,r_{S\setminus M}\right]^2}{|S|(|S|-1)(|S|-2)\cdots(|S|-k+1)} \\ &\leq \frac{(p(1-p))^k}{\binom{L}{k}}\sum_{S\in\mathscr{I}_M,|S|\geq L_M}\mathbb{E}\left[(\partial_M f)\,r_{S\setminus M}\right]^2 \\ &= \frac{(p(1-p))^k}{\binom{L}{k}}\sum_{Q\subseteq W\setminus M,|Q|\geq L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]^2. \end{aligned}$$

Let us now set our first requirement for L_M. This requirement will be $L_M\geq 2k$. Then, each of the numbers $L_M, L_M-1, \ldots, L_M-k+1$ is greater than or equal to $\frac{L_M}{2}$ and their product is at least $L_M^k/2^k$. Therefore, $\binom{L_M}{k}\geq\frac{L_M^k}{2^k\cdot k!}$. The sum $H_k(f)$ defined in (55) and expanded in (57) can now be bounded as follows.

$$\begin{aligned} H_k(f) &\leq (p(1-p))^k\sum_{M\subseteq W,|M|=k}\left(\sum_{Q\subseteq W\setminus M,|Q|<L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]^2\right. \\ &\qquad\left.+\frac{2^k k!}{L_M^k}\sum_{Q\subseteq W\setminus M,|Q|\geq L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]^2\right). \end{aligned} \tag{62}$$

Let us define the random variable g in the following way

$$g = \sum_{Q\subseteq W\setminus M,|Q|<L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]r_Q. \tag{63}$$

The random variable g is the projection of $\partial_M f$ onto the subspace spanned by $\{r_Q\}$ for sets Q of cardinality strictly smaller than L_M-k. The inequality (62) becomes

$$\begin{aligned} &H_k(f) \\ &\leq (p(1-p))^k\sum_{M\subseteq W,|M|=k}\left(\|g\|_2^2+\frac{2^k k!}{L_M^k}\sum_{Q\subseteq W\setminus M,|Q|\geq L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]^2\right). \end{aligned} \tag{64}$$

The second term on the right-hand side of (64) has an excellent coefficient L_M^k in the denominator. We can afford the following generous bound

$$\sum_{Q\subseteq W\setminus M,|Q|\geq L_M-k}\mathbb{E}\left[(\partial_M f)\,r_Q\right]^2\leq\|\partial_M f\|_2^2.$$

The inequality (64) becomes

$$H_k(f) \leq (p(1-p))^k\sum_{M\subseteq W,|M|=k}\left(\|g\|_2^2+\frac{2^k k!}{L_M^k}\|\partial_M f\|_2^2\right). \tag{65}$$

Since the cardinalities are strictly smaller than L_M, we can use Theorem 10. We will take $q'=\frac{3}{2}$. There exists a scalar α_1 such that

$$\|g\|_2^2 \leq e^{\alpha_1 L_M}\|g\|_{3/2}^2. \tag{66}$$

From (66) and Cauchy-Schwarz inequality we now derive

$$\begin{aligned}\|g\|_2^2 &\leq e^{\alpha_1 L_M}\mathbb{E}\left[|g|\cdot|g|^{1/2}\right]^{4/3} \leq e^{\alpha_1 L_M}\left(\mathbb{E}\left[|g|^2\right]^{1/2}\cdot\mathbb{E}\left[|g|\right]^{1/2}\right)^{4/3} \\ &= e^{\alpha_1 L_M}\|g\|_2^{4/3}\cdot\|g\|_1^{2/3}. \end{aligned} \tag{67}$$

Dividing (67) by $\|g\|_2^{4/3}$ and raising to the exponent $\frac{3}{2}$ gives us

$$\|g\|_2 \leq e^{\alpha_2 L_M}\|g\|_1 \leq e^{\alpha_2 L_M}\|\partial_M f\|_1, \tag{68}$$

where $\alpha_2 = \frac{3}{2}\alpha_1$ is a constant that does not depend on n.

Using (68) we transform (65) into

$$H_k(f) \leq (p(1-p))^k \sum_{M\subseteq W,|M|=k}\left(\frac{\theta e^{2\alpha_2 L_M}}{\|\partial_M f\|_2^2/\|\partial_M f\|_1^2}+\frac{2^k k!}{L_M^k}\right)\|\partial_M f\|_2^2. \tag{69}$$

Let us introduce $\hat{\theta} = (p(1-p))^k\cdot\max\{\theta, 2^k\cdot k!\}$, $\alpha = 2\alpha_2$, and

$$B_M = \frac{\|\partial_M f\|_2^2}{\|\partial_M f\|_1^2}. \tag{70}$$

Define the function ψ_M as

$$\psi_M(L) = \frac{e^{\alpha L}}{B_M}+\frac{1}{L^k}, \tag{71}$$

we can re-write (69) as

$$H_k(f) \leq \hat{\theta}\sum_{M\subseteq W,|M|=k}\psi_M(L_M)\cdot\|\partial_M f\|_2^2. \tag{72}$$

We will now choose a convenient L_M. So far, we only had one requirement that L_M must satisfy. The requirement was that L_M must be bounded below by $2k$. The number $2k$ is a constant that does not depend on n.

Observe that if $B_M > e^{6\alpha}$, then the open interval $\left(\frac{\log B_M}{3\alpha}, \frac{\log B_M}{2\alpha}\right)$ is large enough to contain at least one integer. Let B_0 be the real number such that $B > B_0$ implies $B^{\frac{1}{2k}} > \log B$. Let $\hat{B} = \max\{e^{6\alpha}, B_0\}$.

If we assume that $B_M > \hat{B}$, then we choose L_M with

$$L_M = \max\left\{2k, \left\lfloor\frac{\log B_M}{2\alpha}\right\rfloor\right\}.$$

This choice for L_M immediately implies $L_M > \frac{1}{3\alpha}\log B_M$, hence

$$\frac{1}{L_M^k} < \frac{(3\alpha)^k}{\log^k B_M}. \tag{73}$$

The inequality $L_M < \frac{1}{2\alpha}\log B_M$ gives us

$$\frac{e^{\alpha L_M}}{B_M} < \frac{e^{\frac{1}{2}\log B_M}}{B_M} = \frac{1}{\sqrt{B_M}} < \frac{1}{\log^k B_M}. \tag{74}$$

Inequalities (73) and (74) imply that if $B_M > \hat{B}$, then we can choose L_M in such a way that

$$\psi_M(L_M) < \frac{1+(3\alpha)^k}{\log^k B_M}. \tag{75}$$

If $B_M \leq \hat{B}$, then we are going to use a much simpler bound for $\Sigma(M)$ defined in (61). The product of numbers $|S|$, $(|S|-1)$, $\ldots$, $(|S|-k+1)$ in the denominator is at least as big as $k!$. This generous bound is sufficient to cancel $k!$ and we are left with $\Sigma(M) \leq (p(1-p))^k \|\partial_M f\|_2^2$. However, since $B_M \leq \hat{B}$, we have

$$\Sigma(M) \quad \leq \quad (p(1-p))^k \|\partial_M f\|_2^2 \cdot \frac{1+\log^k \hat{B}}{1+\log^k B_M}. \tag{76}$$

Let us now use (70) to replace $\log B_M$ with $2\log \frac{\|\partial_M f\|_2}{\|\partial_M f\|_1}$. In the case $B_M \leq \hat{B}$ we apply (76), while in the case $B_M > \hat{B}$ we apply (72) and (75) to conclude that there exists a constant $C_k \in \mathbb{R}$ such that

$$H_k(f) \quad \leq \quad C_k \cdot \sum_{M\subseteq W, |M|=k} \frac{\|\partial_M f\|_2^2}{1+\left(\log \frac{\|\partial_M f\|_2}{\|\partial_M f\|_1}\right)^k}. \tag{77}$$

We now add (56) and (77) to complete the proof of the theorem. □

Remark. The function $\psi_M(L)$ defined in (71) cannot be bounded by something much better than $\log^{-k} B_M$, as was done in (75). Basic analysis of ψ_M shows that it is convex and increasing for positive L. Its minimum is attained at the solution of the equation $\psi'(L) = 0$, which after the substitutions

$$y = \frac{\alpha}{k+1} L \quad \text{and} \quad x = \frac{\alpha}{k+1}\left(\frac{kB_M}{\alpha}\right)^{1/(k+1)}$$

becomes $ye^y = x$. The function $x \mapsto y(x)$ is not an elementary function, but it is very easy to prove that it is increasing and

$$\lim_{x\to+\infty} \frac{y(x)}{\log x} = 1.$$

3.3. **First-passage percolation on torus.** We now turn to first-passage percolation time f^τ on torus.

Proposition 15. *In the torus model, for every $i \in W_n$ we have*

$$\mathbb{P}(A_i) \quad \leq \quad \frac{1}{p}\mathbb{P}(E_i) \leq \frac{b}{apn^{d-1}}. \tag{78}$$

Proof. The first inequality follows from (38). Observe that all the values $\mathbb{P}(E_j)$ are equal, due to the symmetry of the graph. Therefore,

$$\mathbb{P}(E_i) \quad = \quad \frac{1}{n^d}\sum_{j\in W} \mathbb{P}(E_j) = \frac{1}{n^d}\mathbb{E}\left[\sum_{j\in W} 1_{E_j}\right].$$

The sum $\sum_{j\in W} 1_{E_j}$ is bounded by bn/a because all of the essential edges must be on one geodesic whose length is at most $\frac{bn}{a}$. □

Theorem 11. *Let $M \subseteq W$ be a set with $k \geq 1$ elements. If f^τ is the first passage percolation time on torus, then there exist constants N_0 and $\theta = \theta(k,p)$ such that for all $n \geq N_0$ the following inequality holds*

$$\mathbb{P}(\partial_M f^\tau \neq 0) \leq \frac{\theta}{n^{d-1}}. \tag{79}$$

Proof. We will omit the superscript τ. However, the argument in this proof applies only to the first passage percolation on torus. The inequality is obviuous if $\partial_M f = 0$ almost surely. Assume that $\partial_M f(\omega) \neq 0$ for some ω. There is an ordering $(m_1, \dots, m_k)$ of the set M and a vector $\overrightarrow{\alpha} = (\alpha_2, \dots, \alpha_k) \in \{a,b\}^{k-1}$ such that $\partial_{m_1} f(\sigma_{m_2}^{\alpha_2} \circ \cdots \circ \sigma_{m_k}^{\alpha_k}(\omega)) \neq 0$. Let us denote $\overrightarrow{\nu} = (m_2, \dots, m_k)$. We will now sum over all possible elements m_1 and all possible choices of $\overrightarrow{\nu}$ and $\overrightarrow{\alpha}$.

$$\begin{aligned}
\mathbb{P}(\partial_M f \neq 0) &\leq \sum_{m_1, \overrightarrow{\nu}, \overrightarrow{\alpha}} \mathbb{P}\left(\partial_{m_1} f \circ \sigma_{\overrightarrow{\nu}}^{\overrightarrow{\alpha}} \neq 0\right) \\
&= \sum_{m_1, \overrightarrow{\nu}, \overrightarrow{\alpha}} \sum_{\overrightarrow{\beta} \in \{a,b\}^{k-1}} \mathbb{P}\left(\left\{\partial_{m_1} f \circ \sigma_{\overrightarrow{\nu}}^{\overrightarrow{\alpha}} \neq 0\right\} \cap I_{\overrightarrow{\nu}, \overrightarrow{\beta}}\right). \qquad (80)
\end{aligned}$$

We now use (41) to obtain

$$\begin{aligned}
\mathbb{P}\left(\left\{\partial_{m_1} f \circ \sigma_{\overrightarrow{\nu}}^{\overrightarrow{\alpha}} \neq 0\right\} \cap I_{\overrightarrow{\nu}, \overrightarrow{\beta}}\right) &\leq \frac{\mathbb{P}\left(\overrightarrow{\beta}\right)}{\mathbb{P}\left(\overrightarrow{\alpha}\right)} \mathbb{P}(A_{m_1}) \\
&\leq \left(\frac{\max\{p, 1-p\}}{\min\{p, 1-p\}}\right)^{k-1} \cdot \frac{b}{apn^{d-1}},
\end{aligned}$$

where for the last inequality we used (78). The number of terms in the summations (80) is really large and is exponential in k, because we are summing over all possibilities for m_1, $\overrightarrow{\nu}$, $\overrightarrow{\alpha}$, and $\overrightarrow{\beta}$. However, the number of terms depends only on k. Therefore, the last inequality and (80) can be used to conclude that there exists a scalar θ for which (79) is satisfied. $\square$

Proof of Corollary 1. The denominator in the second term in (6) contains $\frac{\|\partial_M f\|_2}{\|\partial_M f\|_1}$. We find a lower bound for this component using Cauchy's inequality and (79).

$$\begin{aligned}
\|\partial_M f\|_1 &= \|\partial_M f \cdot 1_{\partial_M f \neq 0}\|_1^2 \leq \|\partial_M f\|_2 \cdot \sqrt{\mathbb{P}(\partial_M f \neq 0)} \\
&\leq \|\partial_M f\|_2 \cdot \frac{\sqrt{\theta}}{n^{\frac{d-1}{2}}}.
\end{aligned}$$

Hence, $\log \frac{\|\partial_M f\|_2}{\|\partial_M f\|_1} \geq C_1 \log n$, for some constant C_1.

In the case $k = 2$, we can also bound the first term on the right-hand side of (6) in the following way

$$\begin{aligned}
\sum_{S \subseteq W; 1 \leq |S| < k} (p(1-p))^{|S|} (\mathbb{E}[\partial_S f])^2 &= (p(1-p)) \sum_{i \in W} (\mathbb{E}[\partial_i f])^2 \\
&\leq (p(1-p)) \sum_{i \in W} ((b-a)\mathbb{P}(A_j))^2 \\
&\leq p(1-p)(b-a)^2 n^d \cdot \left(\frac{1}{n^{d-1}}\right)^2 \\
&= p(1-p)(b-a)^2 \cdot \frac{1}{n^{d-2}}.
\end{aligned}$$

In dimension $d \geq 2$ this last quantity can be bounded by a constant, and is, therefore, negligible and dominated by the second term. This proves the first of the two bounds in (7). For the second bound, observe that $\|\partial_M f\|_2^2$ satisfies

$$\begin{aligned}
\|\partial_M f\|_2^2 &= \mathbb{E}\left[|\partial_M f|^2\right] = \mathbb{E}\left[|\partial_M f|^2 \cdot 1_{\{\partial_M f \neq 0\}}\right] \\
&\leq 2^{|M|}(b-a)\mathbb{P}(\partial_M f \neq 0).
\end{aligned}$$

Therefore, the second summation in (6) can be bounded by $\frac{C}{(\log n)^k}\mathbb{E}\left[N_k\right]$, where N_k is the number of sets M of k elements for which $\partial_M f \neq 0$.

It remains to notice that the numerator of each fraction in the summation contains $\|\partial_M f\|_2^2$ which can be bounded from above by $C_2\mathbb{P}(\partial_M f \neq 0)$. Therefore, the right-hand side of (6) is bounded by

$$\frac{C}{\log^2 n}\sum_{|M|=2}\mathbb{P}\left(\partial_M f \neq 0\right) \quad = \quad \frac{C}{\log^2 n}\mathbb{E}\left[\sum_{|M|=2} 1_{\partial_M f\neq 0}\right],$$

and the last summation in the expected value is precisely the random variable N_2. □

As mentioned earlier, we believe that the following conjectures are true, but we do not know how to prove them.

Conjecture 2. *In first passage percolation model, there exists a constant C independent on N such that*

$$\sum_{M\subseteq W_n, |M|=2} \|\partial_M f\|_2^2 \quad \leq \quad C\cdot N.$$

Conjecture 3. *In first passage percolation model, there exists a constant C independent on N such that*

$$\sum_{M\subseteq W_n, |M|=2} \|\partial_M f^{\tau}\|_2^2 \quad \leq \quad C\cdot N.$$

4. EXTREME ENVIRONMENTS

Our next goal is to construct special, extreme, environments on which the derivatives will have very large positive values and very small negative values. These environments will be used to establish the bounds (13) in Theorem 3. In addition, these special environments will be used to show that the sets such as $A_i \setminus E_i$, $A_i \setminus \hat{E}_i$, and $E_i \setminus \hat{A}_i$ are non-empty in general.

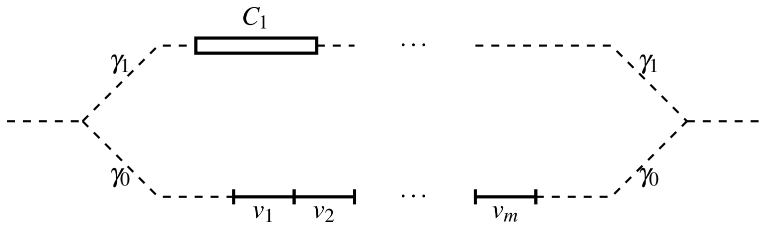
Many of the results in this section involve somewhat lengthy algebraic calculations. Such proofs are presented in the Appendix B.

The results of this section apply to both models: the first-passage percolation between source and sink, and the first-passage percolation time on torus model.

4.1. Environments with overpasses.

We will define one class of environments called *environments with overpasses*. The partial derivatives $\partial_S f$ will have exponentially large values on the environments with overpasses. These environments will be denoted by $\bar{\omega}(m,k)$ for fixed integers m and k that satisfy $1 \leq k \leq m-1$.

Consider the path γ_0 from the source to the sink. Let us identify edges v_1, v_2, …, v_m on the path γ_0. Let L be the number of edges on γ_0 that do not belong to $S = \{v_1, \dots, v_m\}$. Let γ_1 be a path from the source to the sink that is of equal length as γ_0 but does not contain any of the edges v_1, …, v_m. Let $C_1 \subseteq \gamma_1$ be a subset that consists of k edges that are far away from $\{v_1, \dots, v_m\}$. The environment with overpass $\bar{\omega}(m,k)$ assigns the value b to every edge outside of $\gamma_0 \cup \gamma_1$. The value b is also assigned to the k edges that belong to C_1. Every edge on $\gamma_0 \cup \gamma_1 \setminus C_1$ is assigned the value a.



Proposition 16. *For every pair (m,k) that satisfies $1 \leq k \leq m-1$, there exists n_0 such that for $n \geq n_0$, the following holds*

$$\partial_S f(\bar{\omega}(m,k)) \quad = \quad (b-a)\cdot(-1)^{m-1}\cdot\sum_{j=0}^{k-1}(-1)^j\cdot\binom{m}{j}\cdot(k-j). \tag{81}$$

Proof. The proof is in the Appendix B. □

The next proposition implies the bounds (13) in Theorem 3.

Proposition 17. *There is an integer $m_0 > 0$ such that for every $m \geq m_0$ there is an integer n_m and two sets S_m^+ and S_m^- of m edges on the graph $[0,n_m]^d$ that satisfies*

$$\mathbb{P}\left(\partial_{S_m^+} f > \left(\sqrt[4]{3}\right)^m (b-a)\right) > 0 \quad \textit{and} \quad \mathbb{P}\left(\partial_{S_m^-} f < -\left(\sqrt[4]{3}\right)^m (b-a)\right) > 0. \tag{82}$$

Proof. Let m_0 be the integer defined in Proposition 25. We may assume that m_0 is greater than 50, so that the interval $\left[\frac{7m_0}{24}, \frac{m_0}{3}\right]$ contains at least one even and one odd integer. If not, then we just incrase m_0 to be 50.

Let $m \geq m_0$ and let k_e and k_o be one even and one odd integer from $\left[\frac{7m}{24}, \frac{m}{3}\right]$. Let us consider the environments with overpasses $\bar{\omega}_e = \bar{\omega}(m,k_e)$ and $\bar{\omega}_o = \bar{\omega}(m,k_o)$. Let S_e and S_o be the corresponding sets of vertices with respect to which the derivatives are evaluated. Let us define

$$F_e = \frac{(-1)^{m-1}}{b-a}\cdot\partial_{S_e} f\left(\bar{\omega}_e\right) \quad \text{and} \quad F_o = \frac{(-1)^{m-1}}{b-a}\cdot\partial_{S_o} f\left(\bar{\omega}_o\right). \tag{83}$$

We will now take the summation from (81) and group terms into pairs. For F_e we group the term that corresponds to $j=0$ with the term that corresponds to $j=1$. We obtain

$$\begin{aligned} -F_e &= -\sum_{j=0}^{k_e-1}(-1)^j\cdot\binom{m}{j}\cdot(k_e-j) \\ &= \sum_{l=0}^{(k_e-2)/2}\left(\binom{m}{2l+1}(k_e-2l-1)-\binom{m}{2l}(k_e-2l)\right). \end{aligned}$$

Every term in the last summation is non-negative. We obtain a lower bound for $-F_e$ by ignoring all but the last term, and then we use (145).

$$-F_e \quad \geq \quad \binom{m}{k_e-1}-2\binom{m}{k_e-2} > \sqrt[4]{3}^m. \tag{84}$$

Our next task is to obtain a bound for F_o. This time we will group the term that corresponds to $j=1$ with the term that corresponds to $j=2$; the term which corresponds to $j=3$ will be grouped with the term that has $j=4$; and so on.

$$\begin{aligned} F_o &= \sum_{j=0}^{k_o-1}(-1)^j\cdot\binom{m}{j}\cdot(k_o-j) \\ &= k_0+\sum_{l=1}^{\frac{k_o-1}{2}}\left(\binom{m}{2l}(k_o-2l)-\binom{m}{2l-1}(k_o-2l+1)\right). \end{aligned}$$

All terms are positive, hence if we ignore all except for the last one, we obtain the following bound.

$$F_o \geq \binom{m}{k_o-1}-2\binom{m}{k_o-2}>\sqrt[4]{3}^m. \tag{85}$$

The inequalities (84) and (85) together with (83) give us

$$(-1)^m\partial_{S_e}f(\bar{\omega}_e)>\sqrt[4]{3}^m(b-a) \quad\text{and}\quad (-1)^{m-1}\partial_{S_o}f(\bar{\omega}_o)>\sqrt[4]{3}^m(b-a). \tag{86}$$

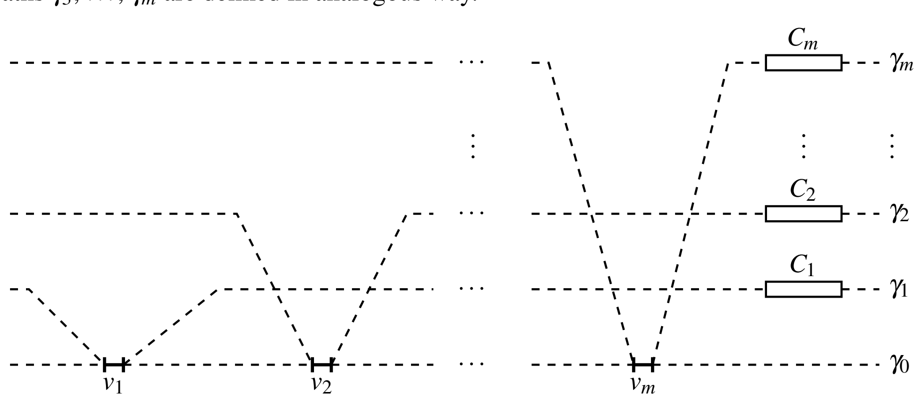
One of the numbers $(-1)^m$ and $(-1)^{m-1}$ is positive and the other is negative. In the case that $(-1)^m=1$ and $(-1)^{m-1}=-1$, we define $(S_m^+,S_m^-,\bar{\omega}_+,\bar{\omega}_-)=(S_e,S_o,\bar{\omega}_e,\bar{\omega}_o)$. If $(-1)^m=-1$ and $(-1)^{m-1}=1$, then we define $(S_m^+,S_m^-,\bar{\omega}_+,\bar{\omega}_-)=(S_o,S_e,\bar{\omega}_o,\bar{\omega}_e)$. In either of the cases, the inequalities (86) turn into

$$\partial_{S_m^+}f(\bar{\omega}_+)>\sqrt[4]{3}^m(b-a) \quad\text{and}\quad \partial_{S_m^-}f(\bar{\omega}_-)<-\sqrt[4]{3}^m(b-a).$$

The last pair of inequalities implies (82). □

4.2. **Environments with valleys.** We will describe another special class of environments $\check{\omega}(m,k)$ that we will call *environments with valleys*. These environments will show that the upper bounds $\partial_{S_2}f\leq(b-a)$, $\partial_{S_3}f\leq 2(b-a)$, and $\partial_{S_4}f\leq 3(b-a)$ cannot be improved for sets S_2, S_3, and S_4 that satisfy $|S_2|=2$, $|S_3|=3$, and $|S_4|=4$.

We will assume that $d\geq 3$. Fix two integers m and k such that $0\leq k\leq m-1$. Consider the path γ_0 from the source to the sink. Let us identify edges v_1, v_2, $\dots$, v_m on the path γ_0. Let L be the number of edges on γ_0 that do not belong to $S=\{v_1,\dots,v_m\}$. Let γ_1 be the path that has only edge v_1 in common with γ_0. We can assume for sufficiently large n, that the path γ_1 has the same length as γ_0, i.e. that in addition to the edge v_1, the path γ_1 has $L+m-1$ edges. Define the path γ_2 to be discjoint with γ_1 and that has only edge v_2 in common with γ_0. We can also assume that there are $L+m-1$ edges in $\gamma_2\setminus\{v_2\}$. The paths γ_3, $\dots$, γ_m are defined in analogous way.



On each of the paths $\gamma_1, \ldots, \gamma_m$ we identify a set of k edges. Let us label those sets as $C_1, \ldots, C_m$. The environment with valleys, $\check{\omega}(m,k)$, is defined in the following way: Every edge outside of the union $\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_m$ has value b; Every edge in $C_1 \cup \cdots \cup C_m$ has value b; Every edge in $\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_m \setminus (C_1 \cup \cdots \cup C_m)$ has value a.

For each fixed m and k, it is possible to choose n large enough, so that the edges v_1, $\ldots$, v_m and the paths $\gamma_1, \ldots, \gamma_m$ can be spaced apart sufficiently such that on each of the environments $\sigma_S^{\vec{\alpha}}(\check{\omega}(m,k))$, the paths $\gamma_0, \gamma_1, \ldots, \gamma_m$ are the only possible geodesics.

Proposition 18. *For every pair (m,k) that satisfies $0 \leq k \leq m-1$, there exists n_0 such that for $n \geq n_0$ the following holds*

$$\begin{aligned} \partial_S f(\check{\omega}(m,k)) \quad &= \quad (b-a) \cdot \left((m-k-1)(m-1) + \sum_{j=1}^{m-k-2} (-1)^j S_{k,j} \right), \qquad (87) \\ \textit{where} \quad S_{k,j} \quad &= \quad (m-k)\binom{m}{j+1} - m\binom{m-1}{j}. \qquad (88) \end{aligned}$$

In the case $k \in \{m-2, m-1\}$, the summation in (87) *should be replaced with* 0*.*

Proof. The proof is in the Appendix B. □

Proposition 19. *For sufficiently large n, there exist sets $S_2, S_3, S_4 \subseteq W$ with $|S_2| = 2$, $|S_3| = 3$, and $|S_4| = 4$ such that*

$$\begin{aligned} &\mathbb{P}(\partial_{S_2} f = b-a) > 0, \quad \mathbb{P}\big(\partial_{S_3} f = 2(b-a)\big) > 0, \\ \textit{and} \quad &\mathbb{P}(\partial_{S_4} f = 3(b-a)) > 0. \qquad (89) \end{aligned}$$

Proof. The first result is obtained when we use (87) with $m = 2$ and $k = 0$; we obtain $\partial_{S_2} f(\check{\omega}(2,0)) = b-a$. The second result is obtained when we use (87) with $m = 3$ and $k = 1$; the corresponding environment derivative is $\partial_{S_3} f(\check{\omega}(3,1)) = 2(b-a)$.

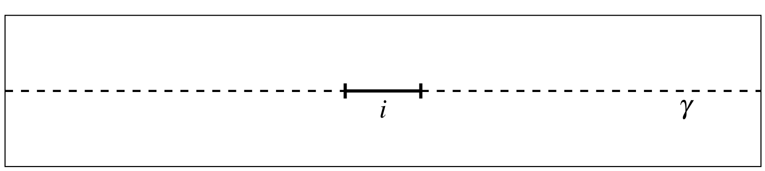
For the third result in (89), we use (87) with $m = 4$ and $k = 2$. The theorem implies $\partial_{S_4} f(\check{\omega}(4,2)) = 3(b-a)$. □

4.3. **Relationship between influential and essential edges.** We established in (30) that $E_i \subseteq A_i$. The following proposition shows that the reverse inclusion does not hold.

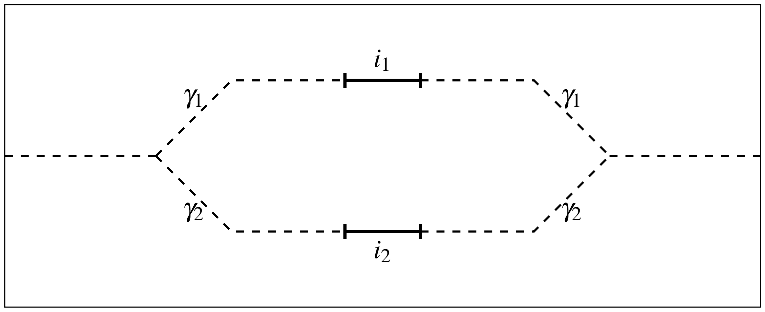
Proposition 20. *There exists n_0 such that for all $n \geq n_0$ there exists an edge i for which the following holds*

$$\begin{aligned} A_i \setminus E_i \quad &\neq \quad \emptyset, \qquad (90) \\ E_i \setminus (A_i \cap \{\omega_i = a\}) \quad &\neq \quad \emptyset. \qquad (91) \end{aligned}$$

Proof. Let us first construct an environment ω in $E_i \cap \{\omega_i = b\}$. This will be a sufficient example to prove (91).



Let us take the straight line γ in the graph. Let us pick one edge on this line γ and call it i. Set ω_k to be b for $k = i$ and for k outside γ. Set ω_l to be a for every edge l on the line γ that is different from i. Then γ is the only geodesic. It passes through i although $\omega_i = b$.



We now construct an environment ω in $A_i \setminus E_i$. Let us pick two paths γ_1 and γ_2 that have the same starting points and the same ending points. However, the paths γ_1 and γ_2 have sections that are reflections of each other, as shown in the picture above. We identify two edges $i_1 \in \gamma_1 \setminus \gamma_2$ and $i_2 \in \gamma_2 \setminus \gamma_1$ that are far away from each other. Consider the environment ω that has the value b on the edges i_1 and i_2 and on every edge outside of $\gamma_1 \cup \gamma_2$. The environment ω has the value a on each edge from $\gamma_1 \cup \gamma_2 \setminus \{i_1, i_2\}$. The edges i_1 and i_2 are influential. However, neither of them is essential, because in the unchanged environment ω, each of γ_1 and γ_2 is a geodesic. □

Proposition 21. *Assume that the real numbers a and b are such that there exist integers k_a and k_b for which $ak_a + bk_b \in (0, b-a)$. Then, there exists an integer n_0 and an edge j such that for all $n \geq n_0$, the following holds*

$$A_j \setminus \hat{E}_j \neq \emptyset; \tag{92}$$

$$E_j \setminus \hat{A}_j \neq \emptyset. \tag{93}$$

Proof. Let us first prove (92). Let us consider two paths γ_1 and γ_2 that have the same starting points and the same ending points, but that contain sections that are sufficiently far away from each other. The passage times are set to b for all edges outside of γ_1 and γ_2. Due our assumptions on a and b, we can make such choices for passage times on discjoint sections of γ_1 and γ_2 such that the difference $T(\gamma_1, \omega) - T(\gamma_2, \omega)$ belongs to the open interval $(0, b-a)$.

Then, let us identify an edge j on the section of γ_1 far away from γ_2 that satisfies $\omega_j = b$. The path γ_2 is the only geodesic on ω and the path γ_1 is the only geodesic on $\sigma_j^a(\omega)$. The edge j is not semi-essential on ω, however it is influential. Hence, $\omega \in A_j \setminus \hat{E}_j$.

The proof for (93) is similar. We can take the same construction that we used in the proof of (92). This time, we identify an edge j' on the section γ_2 that is far away from γ_1 and that satisfies $\omega_{j'} = a$. The path γ_2 is the only geodesic on ω and the path γ_1 is the only geodesic on $\sigma_{j'}^b(\omega)$. Therefore, the edge j' is essential on ω. However, the edge is not very influential, because $\partial_{j'} f(\omega)$ is strictly smaller than $b-a$. □

For a set V of edges, we defined E_V as $\cap_{j \in V} E_j$. Therefore, it makes sense to generalize the concept of essential edge and talk about essential sets of edges. Unfortunately, if we define $A_V = \{\partial_V f \neq 0\}$, the fundamental inclusion $E_j \subseteq A_j$ does not generalize to sets with more than one element. Let us consider the case $V = \{v_1, v_2\}$. The following two propositions imply that $E_V \not\subseteq A_V$ and that $A_V \not\subseteq A_{v_1} \cup A_{v_2}$.

Proposition 22. *For sufficiently large n, there are edges v_1 and v_2 for which the following holds*

$$\{\partial_{v_1} \partial_{v_2} f \neq 0\} \setminus (A_{v_1} \cup A_{v_2}) \neq \emptyset. \tag{94}$$

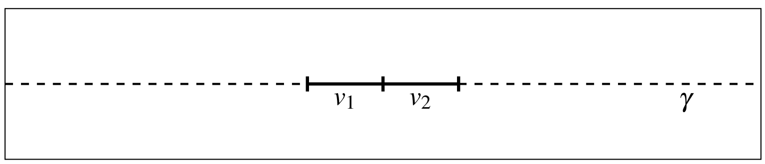
Proof. We will start from the environment with overpass $\bar{\omega}(2,1)$, where we set $m=2$ and $k=1$. We construct ω by copying the environment $\bar{\omega}(2,1)$ and making the following two changes: $\omega_{v_1}=\omega_{v_2}=b$. Then, on the environment ω, the path γ_1 is the only geodesic. Neither v_1 nor v_2 is influential, because turning either ω_{v_1} or ω_{v_2} from b to a would result in both paths γ_0 and γ_1 being the geodesics.

Hence, the environment ω that we constructed satisfies $\omega \notin A_{v_1} \cup A_{v_2}$. However, we know that $\omega \in \{\partial_{v_1}\partial_{v_2} f < 0\} \subseteq \{\partial_{v_1}\partial_{v_2} f \neq 0\}$. □

Proposition 23. *For sufficiently large n, there exists edges v_1 and v_2 such that*

$$(E_{v_1} \cap E_{v_2}) \setminus \{\partial_{v_1}\partial_{v_2} f \neq 0\} \quad \neq \quad \emptyset. \tag{95}$$

Proof. Let us consider a straight line γ and let us identify two edges v_1 and v_2 on the line γ. We will set the environment ω to satisfy $\omega_k = b$ for every $k \notin \gamma$; $\omega_{v_1}=\omega_{v_2}=b$; and $\omega_k = a$ for $k \in \gamma \setminus \{v_1, v_2\}$.



The line γ is the geodesic for sufficiently large n. Each of the edges v_1 and v_2 is essential, hence $\omega \in E_{v_1} \cap E_{v_2}$. Let L be the number of edges on the line γ. The values of the function f at the environments $\sigma_{v_1}^{\alpha_1} \circ \sigma_{v_2}^{\alpha_2}$ for $(\alpha_1, \alpha_2) \in \{a,b\}^2$ are

$$\begin{aligned}
f\left(\sigma_{v_1}^{b} \circ \sigma_{v_2}^{b}(\omega)\right) &= (L-2)a+2b;\\
f\left(\sigma_{v_1}^{a} \circ \sigma_{v_2}^{b}(\omega)\right) &= (L-1)a+b;\\
f\left(\sigma_{v_1}^{b} \circ \sigma_{v_2}^{a}(\omega)\right) &= (L-1)a+b;\\
f\left(\sigma_{v_1}^{a} \circ \sigma_{v_2}^{a}(\omega)\right) &= La.
\end{aligned}$$

Therefore, the value of $\partial_{\{v_1,v_2\}} f(\omega)$ is 0 and $\omega \notin \{\partial_{v_1}\partial_{v_2} f \neq 0\}$. □

5. Almost sure bounds

In this section we prove the Theorem 3. We will first prove the inequalities (11). It suffices to prove the proposition below.

Proposition 24. *Assume φ is a random variable such that for every subset $T \subseteq W$ with k elements we have $\partial_T \varphi \in [L,U]$. Then, the following inequality holds for every subset $S \subseteq W$ with $k+1$ elements.*

$$\partial_S \varphi \in [L-U, U-L]. \tag{96}$$

Proof. Let s be an arbitrary element of S. Let $T = S \setminus \{s\}$.

$$\partial_S \varphi(\omega) \quad = \quad \partial_T \varphi(\sigma_s^b(\omega)) - \partial_T \varphi(\sigma_s^a(\omega)).$$

The result (96) immediately follows from the previous equality. □

Theorem 12. *Let $k \in W$ and let $S \subseteq W$ be a subset with at least two elements. The derivatives of the first-passage percolation time f satisfy the following inequalities for every $\omega \in \Omega$.*

$$\begin{aligned} \partial_k f(\omega) &\in [0, b-a]; && (97) \\ \partial_S f(\omega) &\in [-(b-a), b-a], \quad \text{if } |S| = 2; && (98) \\ |\partial_S f(\omega)| &\leq 2^{|S|-2} \cdot (b-a). && (99) \end{aligned}$$

Proof. The relation (97) is obvious because the function f must increase, and it can increase by at most $b-a$ if one edge changes its passage time from a to b. Let us first observe that for sets S with two elements, the relation (98) and the inequality (99) follow directly from (97) and (96). Observe that if φ is any function, and not just first passage percolation time, then (5) implies $|\partial_G \varphi(\omega)| \leq 2^{|G|} \|\varphi\|_\infty$ for every set G. Assume now that S has at least two elements k and l. Let $G = S \setminus \{k, l\}$.

$$\begin{aligned} |\partial_S f(\omega)| &= |\partial_G (\partial_k \partial_l f(\omega))| \leq 2^{|G|} \cdot \|\partial_k \partial_l f\|_\infty \\ &\leq 2^{|G|} \cdot (b-a). \end{aligned}$$

The proof is complete once we observe that $|G| = |S| - 2$. □

Observe that (99) implies (12). The bounds (13) in Theorem 3 follow from (82) that was proved in Proposition 17.

6. Evaluation of $\mathscr{L}_3$ and $\mathscr{U}_4$

The bound (99) is not sharp. The lower bound can be improved when $|S| = 3$.

Theorem 13. *Let $S \subseteq W$ be a subset with three elements. The first passage percolation time f satisfies the following inequality for every $\omega \in \Omega$*

$$\partial_S f(\omega) \geq -(b-a). \tag{100}$$

Proof. Let $S = \{k, l, m\}$. We will make our notation shorter and write $\sigma^{(\theta_1, \theta_2, \theta_3)}(\omega)$ instead of $\sigma_k^{\theta_1} \circ \sigma_l^{\theta_2} \circ \sigma_m^{\theta_3}(\omega)$ for $(\theta_1, \theta_2, \theta_3) \in \{a, b\}^3$. We will first prove the following implication

$$\sigma^{(a,a,b)}(\omega) \notin E_k \cap E_l \cap E_m^C \implies \partial_S f(\omega) \geq -(b-a) \tag{101}$$

The result (101) will follow from the following two

$$\begin{aligned} \sigma^{(a,a,b)}(\omega) \in E_k^C &\implies \partial_S f(\omega) \geq -(b-a), && (102) \\ \sigma^{(a,a,b)}(\omega) \in E_m &\implies \partial_S f(\omega) \geq -(b-a). && (103) \end{aligned}$$

The first step in proving (102) is to express the derivative $\partial_S f(\omega)$ as

$$\begin{aligned} \partial_S f(\omega) = &\Big(f(\sigma^{(b,b,b)}(\omega)) - f(\sigma^{(b,b,a)}(\omega)) \Big) && (104) \\ &+ \Big(f(\sigma^{(b,a,a)}(\omega)) - f(\sigma^{(a,a,a)}(\omega)) \Big) && (105) \\ &- \Big(f(\sigma^{(b,a,b)}(\omega)) - f(\sigma^{(a,a,b)}(\omega)) \Big) && (106) \\ &- \Big(f(\sigma^{(a,b,b)}(\omega)) - f(\sigma^{(a,b,a)}(\omega)) \Big). && (107) \end{aligned}$$

Assume that $\sigma^{(a,a,b)}(\omega) \in E_k^C$. The Proposition 2 (a) implies that the term (106) is equal to 0. The terms (104) and (105) are non-negative, and the negative term (107) is bounded below by $-(b-a)$, which proves (102). In order to prove (103), we start by expressing $\partial_S f(\omega)$ as

$$\begin{aligned}\partial_S f(\omega) &= \Big(f(\sigma^{(b,b,b)}(\omega)) - f(\sigma^{(b,b,a)}(\omega))\Big) && (108)\\ &\quad + \Big(f(\sigma^{(a,a,b)}(\omega)) - f(\sigma^{(a,a,a)}(\omega))\Big) && (109)\\ &\quad - \Big(f(\sigma^{(b,a,b)}(\omega)) - f(\sigma^{(b,a,a)}(\omega))\Big) && (110)\\ &\quad - \Big(f(\sigma^{(a,b,b)}(\omega)) - f(\sigma^{(a,b,a)}(\omega))\Big). && (111)\end{aligned}$$

Assume that $\sigma^{(a,a,b)}(\omega) \in E_m$. The Proposition 2 (b) implies that the term (109) is equal to $(b-a)$. The term (108) is non-negative. The terms (110) and (111) are negative but bounded below by $-(b-a)$. Therefore, $\partial_S f(\omega)$ is bounded below by $(b-a)-2(b-a) = -(b-a)$. This completes the proof of (103).

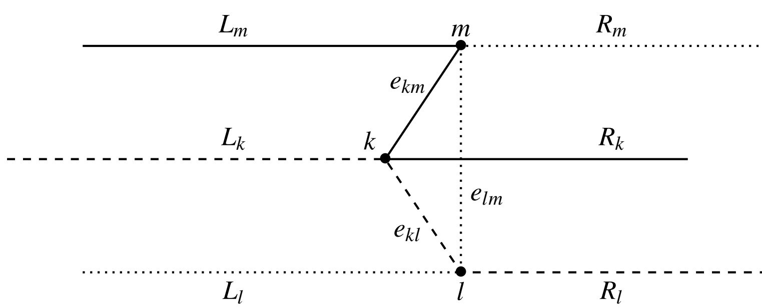
We proved (101) which states that the inequality $\partial_S f(\omega) \geq -(b-a)$ is satisfied unless $\sigma^{(a,a,b)}(\omega)$ is an element of $E_k \cap E_l \cap E_m^C$. The analogous statements hold for $\sigma^{(a,b,a)}(\omega)$ and $\sigma^{(b,a,a)}(\omega)$. Hence, the required bound (100) is proved unless all of the following three inclusions are satisfied

$$\begin{aligned}\sigma^{(a,a,b)}(\omega) &\in E_k \cap E_l \cap E_m^C, && (112)\\ \sigma^{(a,b,a)}(\omega) &\in E_k \cap E_l^C \cap E_m, \quad \text{and} && (113)\\ \sigma^{(b,a,a)}(\omega) &\in E_k^C \cap E_l \cap E_m. && (114)\end{aligned}$$

Hence, it suffices to prove $\partial_S f \geq -(b-a)$ under the conditions (112), (113), and (114). Let γ_{kl} be a geodesic on $\sigma^{(a,a,b)}(\omega)$ that does not contain the edge m. Such geodesic must exist because we assumed that $\sigma^{(a,a,b)}(\omega) \in E_m^C$. The geodesic γ_{kl} must contain both edges k and l. We define the curves γ_{lm} and γ_{km} in analaogous ways. Once the curves γ_{kl}, γ_{lm}, and γ_{km} are fixed, we define the relation $\prec$ on $\{k,l,m\}$. We will write $k \prec l$ if on the curve γ_{kl} the edge k appears before the edge l when moving from the source to the sink. There are two cases:

- Case 1: The relation $\prec$ does not have the minimum in $\{k,l,m\}$;
- Case 2: The relation $\prec$ has the minimum in $\{k,l,m\}$.

Case 1. This case is easier to consider. We will prove that $\partial_S f(\omega) > 0$ which is stronger than the required inequality. We may assume that $k \prec l$, $l \prec m$, and $m \prec k$.



Let us denote by e_{kl} the total passage time between the edges k and l on the geodesic γ_{kl}. We define e_{km} and e_{lm} in analogous way. Let us denote by L_k the total passage time on the geodesic γ_{kl} before the edge k. Let R_l be the total passage time on the geodesic γ_{kl} after the edge l. The numbers L_m, L_l, R_k, and R_m are defined in similar ways. Let us emphasize that none of the previously defined passage times includes the edges k, m, and l. Therefore, the quantities that we defined are the same on the environments $\sigma^{\overrightarrow{\theta}}(\omega)$ for all eight choices $\overrightarrow{\theta} \in \{a,b\}^3$. The following identities hold

$$\begin{aligned} f(\sigma^{(a,a,b)}(\omega)) &= L_k + e_{kl} + R_l + 2a, & (115)\\ f(\sigma^{(a,b,a)}(\omega)) &= L_m + e_{km} + R_k + 2a, & (116)\\ f(\sigma^{(b,a,a)}(\omega)) &\geq f(\sigma^{(a,a,a)}(\omega)), & (117)\\ f(\sigma^{(b,b,b)}(\omega)) &\geq f(\sigma^{(b,b,a)}(\omega)), & (118)\\ f(\sigma^{(b,a,b)}(\omega)) &\leq L_l + a + R_l, & (119)\\ f(\sigma^{(a,b,b)}(\omega)) &\leq L_k + a + R_k. & (120)\end{aligned}$$

The equalities (115) and (116) are due to the definitions of γ_{kl} and γ_{km}. The inequalities (117) and (118) are the consequences of monotonicity. Let us prove (119). On the environment $\sigma^{(b,a,b)}(\omega)$, we can construct a path δ such that $T(\delta,\sigma^{(b,a,b)}(\omega)) = L_l + a + R_l$. Let us identify the section of the curve γ_{lm} before the point l and call it δ_1. It has the passage time L_l. Let us consider the section of the curve γ_{kl} after the point l. This section will be called δ_2. Its passage time is R_l. The curve $\delta = \delta_1 \cup \{l\} \cup \delta_2$ has the passage time $L_l + a + R_l$. The equality (120) is proved in a similar way.

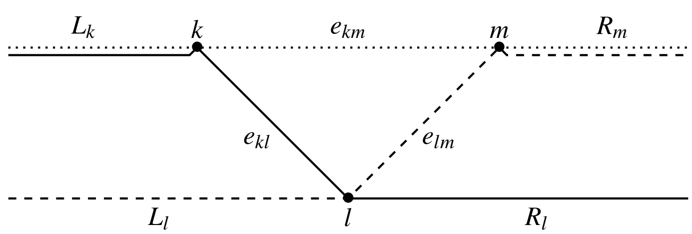
From (115)–(120) we obtain

$$\begin{aligned} \partial_S f(\omega) &\geq (L_k + e_{kl} + R_l + 2a) + (L_m + e_{km} + R_k + 2a)\\ &\quad -(L_l + a + R_l) - (L_k + a + R_k)\\ &= e_{kl} + L_m + e_{km} + 2a - L_l. \qquad (121)\end{aligned}$$

Let us consider the geodesic γ_{lm} on the environment $\sigma^{(b,a,a)}(\omega)$. Let us consider the curve ξ that passes through m and consists of the left part of γ_{km} and the right part of γ_{lm}. The curve ξ is not a geodesic on $\sigma^{(b,a,a)}(\omega)$ because of (114). Hence, $L_m + a + R_m > L_l + e_{lm} + R_m + 2a$. The last inequality is equivalent to $L_m - L_l > e_{lm} + a > 0$. The inequality (121) turns into $\partial_S f(\omega) \geq e_{kl} + e_{km} + 3a > 3a > -(b-a)$. This finishes the proof in Case 1.

Remark. We proved that $\partial_S f > 3a$. If the number $3a$ were larger than $2(b-a)$, which is the maximal possible value for $\partial_S f$, then Case 1 would not be possible and Case 2 would be the only one worth considering.

Case 2. We may assume that k is the minimum, i.e. $k \prec l$ and $k \prec m$. Without loss of generality, we may assume that $l \prec m$.



The sections of the curves γ_{kl} and γ_{km} before the edge k must have equal passage times. Let us denote by L_k the common passage time of these sections. We may modify one of the curves γ_{kl} and γ_{km} in such a way that the sections before k actually coincide. In a similar way, the passage times after the edge m on the curves γ_{km} and γ_{lm} are equal. We will denote these passage times by R_m. We define L_l as the passage time over the curve γ_{lm} before the edge l; R_l the passage time over γ_{kl} after l. We define e_{kl}, e_{lm}, and e_{km} as passage times over the open intervals (k,l), (l,m), and (k,m) on the curves γ_{kl}, γ_{lm}, and γ_{km}, respectively. Let us define the real number θ in such a way that the following holds

$$e_{km} = e_{kl} + e_{lm} + a + \theta. \tag{122}$$

Such θ is unique as it is a solution to a simple linear equation. We don't know whether θ is positive or negative. But we do know that θ satisfies the inequality $\theta \le b-a$. This follows from the fact that on $\sigma^{(a,b,a)}(\omega)$, the minimal passage time is over the curve γ_{km}. This passage time is $L_k + e_{km} + R_m + 2a$ and must be smaller than or equal to the passage time $L_k + e_{kl} + e_{lm} + R_m + 2a + b$ over the modified curve in which the segment (k,m) is replaced with $(k,l) \cup \{l\} \cup (l,m)$. Let us define the real numbers θ_L and θ_R with the following two identities

$$L_l = L_k + e_{kl} + a + \theta_L, \tag{123}$$

$$R_l = R_m + e_{lm} + a + \theta_R. \tag{124}$$

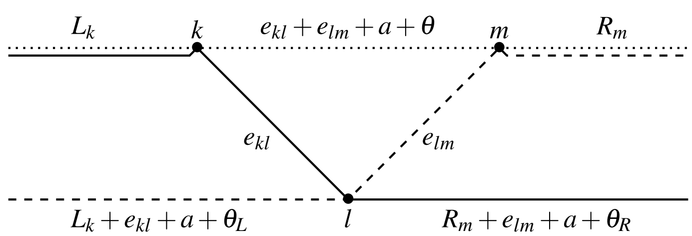
The numbers θ_L and θ_R must belong to the interval $(0, b-a]$. Let us prove that $\theta_L \in (0, b-a)$. We need to prove $\theta_L > 0$ and $\theta_L \le b-a$. The inequality $\theta_L > 0$ follows from the fact that on $\sigma^{(a,a,b)}(\omega)$, every geodesic must go through k and the path γ_{kl} is a geodesic. If we take the section of this geodesic before the edge l and replace it with the corresponding section of γ_{lm}, then we will get something that is not a geodesic because of (112). The change of the passage time must satisfy

$$0 < L_l - L_k - a - e_{kl} = \theta_L.$$

Let us now prove that $\theta_L \le b-a$. Consider the environment $\sigma^{(b,a,a)}(\omega)$. The curve γ_{lm} is a geodesic. When we replace the section before l with the corresponding section of γ_{kl}, the passage time may only increase. The change in the passage time is

$$\begin{aligned} 0 &\le -L_l + L_k + b + e_{kl} \\ &= -(L_k + e_{kl} + a + \theta_L) + L_k + b + e_{kl}, \end{aligned}$$

hence $\theta_L \le b-a$. In an analogous way we prove that $\theta_R \in (0, b-a]$.



Since γ_{kl}, γ_{lm}, and γ_{km} are geodesics on $\sigma^{(a,a,b)}(\omega)$, $\sigma^{(b,a,a)}(\omega)$, and $\sigma^{(a,b,a)}(\omega)$, respectively, we obtain

$$f(\sigma^{(a,a,b)}(\omega)) = L_k + R_m + e_{kl} + e_{lm} + 3a + \theta_R, \tag{125}$$

$$f(\sigma^{(a,b,a)}(\omega)) = L_k + R_m + e_{kl} + e_{lm} + 3a + \theta, \tag{126}$$

$$f(\sigma^{(b,a,a)}(\omega)) = L_k + R_m + e_{kl} + e_{lm} + 3a + \theta_L. \tag{127}$$

Due to monotonicity we have

$$f(\sigma^{(b,b,b)}(\omega)) \ge f(\sigma^{(a,b,b)}(\omega)). \tag{128}$$

Let us consider the environment $\sigma^{(b,a,b)}(\omega)$ and the curve ζ that passes through l and not through k and m. The section of ζ before l coincides with γ_{lm} and the section after

l coincides with γ_{kl}. The passage time over ζ is greater than or equal than the minimal passage time, hence

$$f(\sigma^{(b,a,b)}(\omega)) \quad \leq \quad L_k + R_m + e_{kl} + e_{lm} + 3a + \theta_L + \theta_R. \tag{129}$$

The minimal passage time $f(\sigma^{(b,b,a)}(\omega))$ is smaller than or equal to the minimum of the passage times over the curves γ_{lm} and γ_{km}, hence

$$f(\sigma^{(b,b,a)}(\omega)) \quad \leq \quad L_k + R_m + e_{kl} + e_{lm} + 2a + b + \min\{\theta_L, \theta\}. \tag{130}$$

Finally, let us consider the environment $\sigma^{(a,a,a)}(\omega)$. By considering the passage time over the curve γ_{km}, we obtain

$$f(\sigma^{(a,a,a)}(\omega)) \quad \leq \quad L_k + R_m + e_{kl} + e_{lm} + 3a + \theta. \tag{131}$$

Let us consider the curve γ'_{lm} in which we add the vertex l and replace the section (k,m) with the sections (k,l) and (l,m) of the curves γ_{kl} and γ_{lm}. The passage time over γ'_{lm} gives us

$$f(\sigma^{(a,a,a)}(\omega)) \quad \leq \quad L_k + R_m + e_{kl} + e_{lm} + 3a. \tag{132}$$

The inequalities (131) and (132) imply

$$f(\sigma^{(a,a,a)}(\omega)) \quad \leq \quad L_k + R_m + e_{kl} + e_{lm} + 3a + \min\{\theta, 0\}. \tag{133}$$

We now use (125)–(130) and (133) to find the lower bound on $\partial_S f(\omega)$. Observe that $R_k + R_m + e_{kl} + e_{lm}$ appears equally many times with sign $+$ as with sign $-$. We can ignore these terms. Hence,

$$\partial_S f(\omega) \quad \geq \quad a - b + \theta - \min\{\theta_L, \theta\} - \min\{\theta, 0\}. \tag{134}$$

Define $F(\theta, \theta_L) = \theta - \min\{\theta_L, \theta\} - \min\{\theta, 0\}$. It sufices to prove that $F(\theta, \theta_L) \geq 0$. There are two cases: $\theta \geq 0$ and $\theta < 0$. If $\theta \geq 0$, then $\min\{\theta, 0\} = 0$ and $F(\theta, \theta_L) = \theta - \min\{\theta_L, \theta\} \geq 0$. If $\theta < 0$, then from $\theta_L > 0$ we have $\min\{\theta, \theta_L\} = \theta$, and $F(\theta, \theta_L) = \theta - \theta - \theta = -\theta > 0$. This completes the proof of Case 2, which was the only remaining case that we needed to consider. $\square$

Theorem 14. *The first four elements of the sequence $(\mathscr{U}_n)$ and the first three elements of the sequence $(\mathscr{L}_n)$ are*

$$(\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3, \mathscr{U}_4) \quad = \quad (1, 1, 2, 3); \tag{135}$$

$$(\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \quad = \quad (0, -1, -1). \tag{136}$$

Proof. The inequalities $\mathscr{U}_1 \leq 1$ and $\mathscr{L}_1 \geq 0$ follow from (97). The bound $\mathscr{U}_1 \geq 1$ can be proved by constructing an environment ω for which there is an edge k such that $\partial_k f(\omega) = b - a$. This is easy to do: ω that assigns b to every edge satisfies the required property. The bound $\mathscr{L}_1 \leq 0$ is equally easy to prove – an environment ω that assigns a to every edge satisfies $\partial_k f(\omega) = 0$ for every k.

The inequalities $\mathscr{U}_2 \leq 1$ and $\mathscr{L}_2 \geq -1$ follow from (98). Proposition 16 with $m = 2$ and $k = 1$ implies that $\partial_S f(\bar{\omega}(2,1)) = -(b-a)$, hence $\mathscr{L}_2 \leq -1$. Proposition 19 implies that $\mathscr{U}_2 \geq 1$.

In addition, Proposition 19 also implies $\mathscr{U}_3 \geq 2$ and $\mathscr{U}_4 \geq 3$. The first inequality in (11) implies that $\mathscr{U}_3 \leq \mathscr{U}_2 - \mathscr{L}_2 = 2$.

From (100) we have $\mathscr{L}_3 \geq -1$. Proposition 16 with $m = 3$ and $k = 2$ implies that $\partial_S f(\bar{\omega}(3,2)) = -(b-a)$, which implies $\mathscr{L}_3 \leq -1$. Finally, we again use the inequality (11) to obtain $\mathscr{U}_4 \leq \mathscr{U}_3 - \mathscr{L}_3 = 2 - (-1) = 3$. $\square$

Theorem 15. *The number $\mathscr{L}_4$ satisfies $\mathscr{L}_4 \leq -2$.*

Proof. It suffices to prove that for large n, there is always an $\omega \in \{a,b\}^{W_n}$ and a set $S_4 \subseteq W_n$ with four elements such that $\partial_{S_4} f(\omega) = -2(b-a)$. Let $S_4 = \{v_1, v_2, v_3, v_4\}$, and assume that v_4 is very far from v_1, v_2, and v_3. We will assume that there is a path γ_{123} through v_1, v_2, and v_3, and another distant path γ_4 through v_4. We will assume that the passage time through $\gamma_{123} \setminus \{v_1, v_2, v_3\}$ is L and that the passage time through $\gamma_4 \setminus \{v_4\}$ is $L+a+b$. It is straightforward to verify that $\partial_{S_4}(\omega) = -2(b-a)$. □

Appendix A. Idempotent functions

Proof of Proposition 3. If $x \in \psi(R)$, then there is $z \in R$ such that $x = \psi(z)$. We now have $\psi(x) = \psi(\psi(z)) = \psi(z) = x$, hence $x = \psi(x)$. This proves $\psi(R) \subseteq \{x \in R : \psi(x) = x\}$. The reverse inclusion is obvious. □

Proof of Proposition 4. The equality (23) will follow from the following four inclusions

$$\begin{aligned} & \psi\left(\psi^{-1}(Q) \cap \xi(R)\right) \subseteq \psi(\psi^{-1}(Q)) \subseteq Q \cap \psi(R) \subseteq \psi^{-1}(Q) \cap \psi(R) \\ \subseteq \quad & \psi\left(\psi^{-1}(Q) \cap \xi(R)\right). \end{aligned} \tag{137}$$

The first two of the inclusions hold for all functions ψ and ξ, not just the idempotent ones. Indeed, $\psi^{-1}(Q) \cap \xi(R) \subseteq \psi^{-1}(Q)$ implies the first inclusion; the relation $\psi(\psi^{-1}(Q)) \subseteq Q$ holds for all ψ and Q, while $\psi^{-1}(Q) \subseteq R$ implies $\psi(\psi^{-1}(Q)) \subseteq \psi(R)$.

Let us now prove the third inclusion. Assume that $x \in Q \cap \psi(R)$. We need to prove that $x \in \psi^{-1}(Q)$. From (22) we have $x = \psi(x)$ which is sufficient to conclude that $\psi(x) \in Q$.

We now prove the fourth inclusion in (137), i.e.

$$\psi^{-1}(Q) \cap \psi(R) \quad \subseteq \quad \psi\left(\psi^{-1}(Q) \cap \xi(R)\right).$$

Let $x \in \psi^{-1}(Q) \cap \psi(R)$. We need to prove that there exists an element y of the set $\psi^{-1}(Q) \cap \xi(R)$ such that $\psi(y) = x$. We will prove that we may take $y = \xi(x)$. We must prove that $\psi(y) = x$, $y \in \psi^{-1}(Q)$, and $y \in \xi(R)$. The last assertion, that $y \in \xi(R)$, is obvious because of our choice $y = \xi(x)$. Since $x \in \psi(R)$, the equality (22) implies $x = \psi(x) = \psi(\xi(x)) = \psi(y)$, which implies the first assertion that $\psi(y) = x$. The second assertion, $y \in \psi^{-1}(Q)$, follows from $\psi(y) = \psi(\xi(x)) = \psi(x) \in Q$, which holds due to our assumption $x \in \psi^{-1}(Q)$. □

Proof of Proposition 5. Since $\psi^a(R) \cup \psi^b(R) = R$, it suffices to prove the following four inclusions

$$E \cap \psi^a(R) \quad \subseteq \quad (\psi^a)^{-1}(E), \tag{138}$$

$$E \cap \psi^b(R) \quad \subseteq \quad (\psi^a)^{-1}(E), \tag{139}$$

$$(\psi^b)^{-1}(\hat{E}) \cap \psi^a(R) \quad \subseteq \quad \hat{E}, \tag{140}$$

$$(\psi^b)^{-1}(\hat{E}) \cap \psi^b(R) \quad \subseteq \quad \hat{E}. \tag{141}$$

We will first apply (23) to $\psi = \psi^a$ and $Q = E$. We won't use the first equality, so we don't have to make a choice for ξ.

$$E \cap \psi^a(R) = (\psi^a)^{-1}(E) \cap \psi^a(R) \subseteq (\psi^a)^{-1}(E),$$

which implies (138). Now we use our assumptions $E \subseteq \hat{E}$ and $(\psi^b)^{-1}(\hat{E}) \subseteq (\psi^a)^{-1}(E)$. We apply (23) to $\psi = \psi^b$ and $Q = \hat{E}$.

$$E \cap \psi^b(R) \subseteq \hat{E} \cap \psi^b(R) = (\psi^b)^{-1}(\hat{E}) \cap \psi^b(R) \subseteq (\psi^b)^{-1}(\hat{E}) \subseteq (\psi^a)^{-1}(E).$$

The last inclusion implies (139).

We now prove (141). We apply (23) to $\psi = \psi^b$ and $Q = \hat{E}$.

$$(\psi^b)^{-1}(\hat{E}) \cap \psi^b(R) = \hat{E} \cap \psi^b(R) \subseteq \hat{E}.$$

For the proof of (140), we use the assumptions $E \subseteq \hat{E}$ and $(\psi^b)^{-1}(\hat{E}) \subseteq (\psi^a)^{-1}(E)$ and apply (23) to $\psi = \psi^a$ and $Q = E$.

$$(\psi^b)^{-1}(\hat{E}) \cap \psi^a(R) \subseteq (\psi^a)^{-1}(E) \cap \psi^a(R) = E \cap (\psi^a)^{-1}(R) \subseteq E \subseteq \hat{E}.$$

This completes the proof of (140). □

APPENDIX B. EVALUATIONS OF DERIVATIVES IN EXTREME ENVIRONMENTS

Proof of Proposition 16. Let L be the number of edges in $\gamma_0 \setminus S$. Denote $\overrightarrow{s} = (v_1, \dots, v_m)$. The passage time $T(\gamma_1, \sigma_{\overrightarrow{s}}^{\overrightarrow{\alpha}}(\bar{\omega}(m,k)))$ does not depend on $\overrightarrow{\alpha} \in \{a,b\}^m$ and satisfies

$$T\left(\gamma_1, \sigma_{\overrightarrow{s}}^{\overrightarrow{\alpha}}(\bar{\omega}(m,k))\right) \quad = \quad La + m(b-a) - (m-k)(b-a). \tag{142}$$

If $N_a(\overrightarrow{\alpha})$ is the number of times that a appears in $\overrightarrow{\alpha}$, then

$$T\left(\gamma_0, \sigma_{\overrightarrow{s}}^{\overrightarrow{\alpha}}(\bar{\omega}(m,k))\right) \quad = \quad La + m(b-a) - N_a(\overrightarrow{\alpha})(b-a). \tag{143}$$

The equations (142) and (143) imply

$$f\left(\sigma_{\overrightarrow{s}}^{\overrightarrow{\alpha}}(\bar{\omega}(m,k))\right) \quad = \quad La + m(b-a) - (b-a)\max\{N_a(\overrightarrow{\alpha}), m-k\}. \tag{144}$$

Using (144) we now derive the formula for $\partial_S f(\bar{\omega}(m,k))$. Notice that due to (5) all terms $La + m(b-a)$ cancel. Hence, we can ignore them in the calculation. The formula (5) becomes

$$\partial_S f(\bar{\omega}(m,k)) \quad = \quad -(b-a)\left(\sum_{i=0}^{m-k}(-1)^i\binom{m}{i}(m-k) + \sum_{i=m-k+1}^{m}(-1)^i i\binom{m}{i}\right).$$

Since $\sum_{i=0}^{m}(-1)^i\binom{m}{i} = 0$, the last sum turns into the following

$$\begin{aligned}\partial_S f(\bar{\omega}(m,k)) \quad &= \quad -(b-a)\left(-\sum_{i=m-k+1}^{m}(-1)^i\binom{m}{i}(m-k) + \sum_{i=m-k+1}^{m}(-1)^i i\binom{m}{i}\right)\\ &= \quad -(b-a)\sum_{i=m-k+1}^{m}(-1)^i\,(i-(m-k))\binom{m}{i}.\end{aligned}$$

After the substitution $j = m - i$, the last equality becomes (81). □

Proposition 25. *If $m \geq 3k$, the sequence $\left(\binom{m}{j}(k-j)\right)_{j=0}^{k-1}$ is increasing and non-negative. There exists m_0, such that for $m \geq m_0$ and $k \in [\frac{7m}{24}, \frac{m}{3}]$, the difference of the last two terms is bounded below by $3^{m/4}$, i.e.*

$$\binom{m}{k-1} - 2\binom{m}{k-2} \quad \geq \quad 3^{\frac{m}{4}}. \tag{145}$$

Proof. It is clear that the terms are non-negative because $j \leq k-1$, hence $k-j \geq 1$. Assume that $m \geq 3k$ and that $j \leq k-2$. We will prove that the difference of the term $j+1$ and the term j is positive.

$$\begin{aligned}
& \binom{m}{j+1}(k-j-1) - \binom{m}{j}(k-j) \\
= \quad & \frac{m!}{(j+1)!(m-j-1)!}(k-j-1) - \frac{m!}{j!(m-j)!}(k-j) \\
= \quad & \frac{m!}{(j+1)!(m-j)!}(k-j-1)\left((m-j) - \left(1+\frac{1}{k-j-1}\right)(j+1)\right) \\
\geq \quad & \frac{m!}{(j+1)!(m-j)!}(k-j-1)(m-j-2(j+1)).
\end{aligned}$$

We now use that $m \geq 3k$ and $k \geq j+2$ to obtain

$$m-j-2(j+1) \geq 3(j+2)-3j-2 = 4 > 0.$$

Let us prove (145).

$$\begin{aligned}
\binom{m}{k-1} - 2\binom{m}{k-2} \quad &= \quad \frac{m!}{(k-1)!(m-k+1)!} - 2\frac{m!}{(k-2)!(m-k+2)!} \\
&= \quad \frac{m!}{(k-1)!(m-k+2)!}((m-k+2)-2(k-1)) \\
&= \quad \frac{m!}{(k-1)!(m-k+2)!}(m-3k+4) \\
&\geq \quad \frac{4m!}{(k-1)!(m-k+2)!} = \frac{4}{m+1}\binom{m+1}{k-1} \\
&= \quad \frac{4}{m+1} \cdot \prod_{j=1}^{k-1} \frac{m+2-j}{k-j}.
\end{aligned}$$

For each j we have $\frac{m+2-j}{k-j} \geq 3$ because $3k \leq m$. Hence,

$$\binom{m}{k-1} - 2\binom{m}{k-2} \quad \geq \quad \frac{4}{m+1} \cdot 3^{k-1} > \frac{1}{m+1} \cdot 3^k.$$

Since $k \geq \frac{7m}{24}$, we have

$$\binom{m}{k-1} - 2\binom{m}{k-2} \quad > \quad \frac{3^{\frac{m}{24}}}{m+1} 3^{\frac{m}{4}}.$$

For sufficienlty large m_0, we have $3^{m/24} > m+1$ if $m \geq m_0$, which implies (145). □

Proof of Proposition 18. Let $\overrightarrow{s} = (v_1, \ldots, v_m)$. Recall that for $\alpha \in \{a,b\}^m$, we use $N_a(\overrightarrow{\alpha})$ to denote the number of components that are equal to a. The environment $\check{\omega}(m,k)$ is simple enough that $f(\sigma_{\overrightarrow{s}}^{\overrightarrow{\alpha}}(\check{\omega}(m,k)))$ depends only on $N_a(\overrightarrow{\alpha})$.

$$\begin{aligned}
f(\sigma_{\overrightarrow{s}}^{\overrightarrow{\alpha}}(\check{\omega}(m,k))) \quad &= \quad La + m(b-a) \\
&\quad -(b-a) \cdot \begin{cases} m-k-1, & \text{if } \overrightarrow{\alpha} = (b,b,\ldots,b) \\ \max\{m-k, N_a(\overrightarrow{\alpha})\}, & \text{if } \overrightarrow{\alpha} \neq (b,b,\ldots,b). \end{cases}
\end{aligned}$$

We use (5) to evaluate $\partial_S f(\breve{\omega}(m,k))$. Since each term contains $La+m(b-a)$, these components cancel each other. The derivative $\partial_S f(\breve{\omega}(m,k))$ becomes

$$\begin{aligned}\partial_S f(\breve{\omega}(m,k)) &= -(b-a)\left(m-k-1+(m-k)\sum_{i=1}^{m-k}(-1)^i\binom{m}{i}\right.\\ &\qquad\left.+\sum_{i=m-k+1}^{m}(-1)^i i\binom{m}{i}\right). \end{aligned} \tag{146}$$

The last sum can be simplified as

$$\begin{aligned}\sum_{i=m-k+1}^{m}(-1)^i i\binom{m}{i} &= -m\sum_{i=m-k+1}^{m}(-1)^{i-1}\binom{m-1}{i-1} = -m\sum_{i=m-k}^{m-1}(-1)^i\binom{m-1}{i}\\ &= m\sum_{i=0}^{m-k-1}(-1)^i\binom{m-1}{i}.\end{aligned}$$

We now substitute the last identity into (146) and extract the first and the last term from each of the summations. We obtain

$$\begin{aligned}\partial_S f(\breve{\omega}(m,k)) &= -(b-a)\Big(m-k-1-(m-k)\cdot m+m\\ &\quad +(-1)^{m-k}\cdot\left((m-k)\binom{m}{m-k}-m\binom{m-1}{m-k-1}\right) &(147)\\ &\quad +(m-k)\sum_{i=2}^{m-k-1}(-1)^i\binom{m}{i}+m\sum_{i=1}^{m-k-2}(-1)^i\binom{m-1}{i}\Big). &(148)\end{aligned}$$

The term (147) is equal to 0. After we apply the substitution $j=i-1$ in the first summation of (148) and the substitution $j=i$ in the second, the last equality becomes (87). □

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