46-th International Mathematical Olympiad

Mérida, Mexico, July 8–19, 2005

First Day – July 13

1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent. *(Romania)*

2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \dots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence. *(Netherlands)*

3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$
(South Korea)

Second Day – July 14

4. Consider the sequence $a_1, a_2, \dots$ defined by
$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \dots).$$
Determine all positive integers that are relatively prime to every term of the sequence. *(Poland)*

5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P. *(Poland)*

6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems. *(Romania)*

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