12-th Iberoamerican Mathematical Olympiad  
Guadalajara, Mexico, September 14–21, 1997

First Day

1. A real number $r \geq 1$ has the following property: For any positive integers $m$ and $n$, $m$ divides $n$ if and only if $[mr]$ divides $[nr]$. Prove that $r$ is an integer.

2. A circle centered at the incenter $I$ of a triangle $ABC$ meets all three sides of the triangle: side $BC$ at $D$ and $P$ (with $D$ nearer to $B$), side $CA$ at $E$ and $Q$ (with $E$ nearer to $C$), and side $AB$ at $F$ and $R$ (with $F$ nearer to $A$). The diagonals of the quadrilaterals $EQFR$, $FRDP$, and $DPEQ$ meet at $S$, $T$, and $U$, respectively. Show that the circumcircles of the triangles $FRT$, $DPU$ and $EQS$ have a single point in common.

3. For an integer $n \geq 2$, let $D_n$ be the set of points $(x, y)$ of the plane with integer coordinates such that $-n \leq x, y \leq n$.

(a) Each of the points of $D_n$ is colored with one of three given colors. Prove that there always exist two points of $D_n$ of the same color such that the line passing through them contains no other point of $D_n$.

(b) Give an example of a coloring of points of $D_n$ with four colors in such a manner that if a line contains exactly two points of $D_n$, then these two points have different colors.

Second Day

4. Let $n$ be a positive integer. Consider the sum $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ for any $2n$ numbers $a_i, b_i$ taking only the values 0 and 1. Denote by $I(n)$ the number of $2n$-tuples $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ for which this sum is odd, and by $P(n)$ the number of those for which this sum is even. Prove that

$$\frac{P(n)}{I(n)} = \frac{2^n + 1}{2^n - 1}.$$

5. In a triangle $ABC$, $AE$ and $BF$ are altitudes and $H$ the orthocenter. The line symmetric to $AE$ with respect to the bisector of $\angle A$ and the line symmetric to $BF$ with respect to the bisector of $\angle B$ intersect at a point $O$. The lines $AE$ and $AO$ meet the circumcircle of $\triangle ABC$ again at $M$ and $N$, respectively. The lines $BC$ and $HN$ meet at $P$, $BC$ and $OM$ at $R$, and $HR$ and $OP$ at $S$. Prove that $AHSO$ is a parallelogram.

6. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_{1997}\}$ be a set of 1997 points inside the unit circle with center at $P_1$. For each $k = 1, 2, \ldots, 1997$, let $x_k$ be the distance from $P_k$ to the closest point in $\mathcal{P}$ different from $P_k$. Prove that

$$x_1^2 + x_2^2 + \cdots + x_{1997}^2 \leq 9.$$